DISORDERED SYSTEMS DAYS AT KING'S COLLEGE LONDON

Random lasers as complex disordered systems: a spin-glass-like theory for amplified mode-locking waves in random media

King's College London
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Dario Ballarini, Milena De Giorgi, Luisa De Marco, Giovani Lerario, Daniele Sanvitto.

Outline

- Standard and random lasers
- Statistical physics approach to laser physics
  Theory for ultrafast mode-locked multimode lasers
  (order, closed cavity)
  Theory for random lasers: a mode-locked spin-glass theory
  (disorder, open cavity)
- The narrow-band solution, phase diagrams, replica symmetry breaking, a new overlap: intensity fluctuation overlap
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps
- In between theory and experiment: a mode-locking model
  Monte Carlo dynamics simulation with exchange Monte Carlo,
  GPU parallel computing
- Power distribution among modes in the glassy light regime: condensation vs equipartition at high pumping
- Outlook (work in progress)

• Standard and random lasers

- 1953-1955: **Charles H. Townes, Nikolay Basov, Aleksandr Prokhorov**: Microwave Amplification by Stimulated Emission of Radiation – MASER. They implemented continuous output, gain media with multienergy level atoms, optical pumping for population inversion.
  - Nobel Prize in Physics 1964, "for fundamental work in the field of quantum electronics, which has led to the construction of oscillators and amplifiers based on the maser–laser principle”

- 1958: *Infrared and Optical Masers*, Arthur L. Schawlow and Charles H. Townes:
  - Optical Maser = **Laser** :-) by Gordon Gould (1957, 1959).
  - Also terms Xaser, Uaser, …, Raser…. :-/

- Laser can be single mode or **multimode**,
  continuous wave (laser pointer) or **pulsed** (“ns”, “ps”, “fs”), ...

Ultrafast multimode
Laser

Two essential components

- Cavity — Coherent feedback
- Gain medium — Amplification by Stimulated Emission
- Saturable absorber — Passive mode-locking

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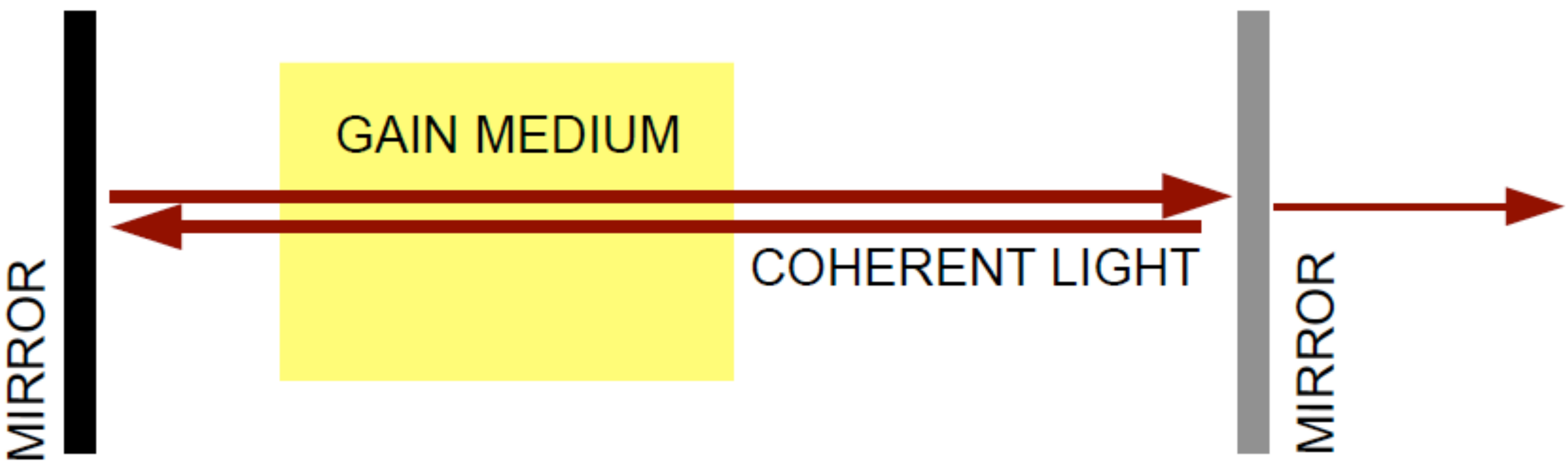
Nobel prize in physics 1964



Ultrafast Multimode Laser

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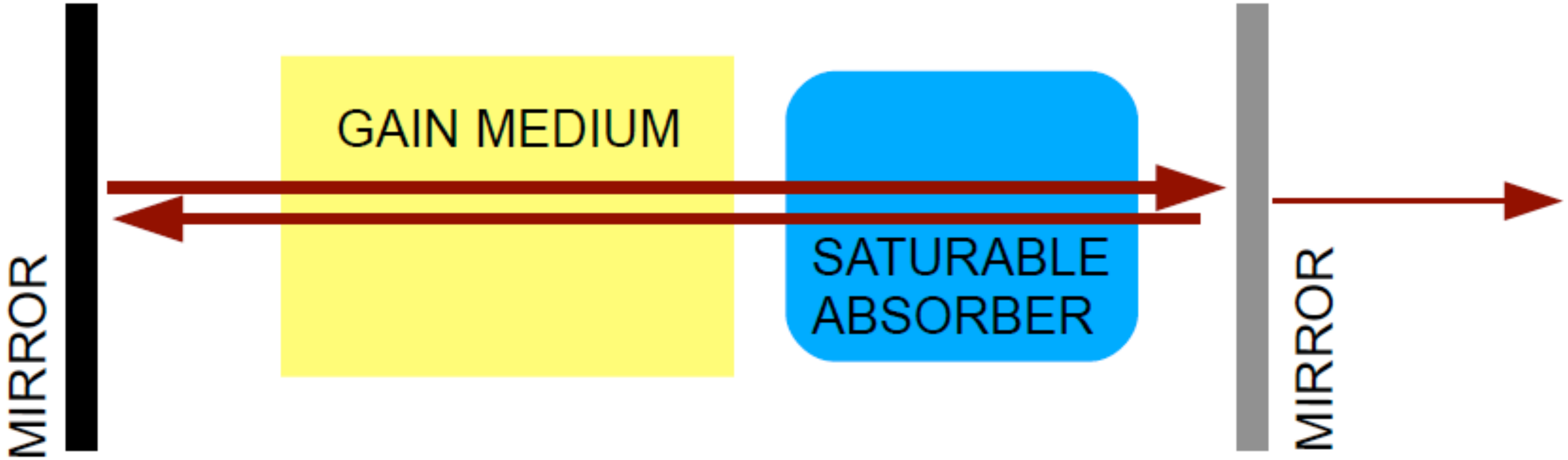
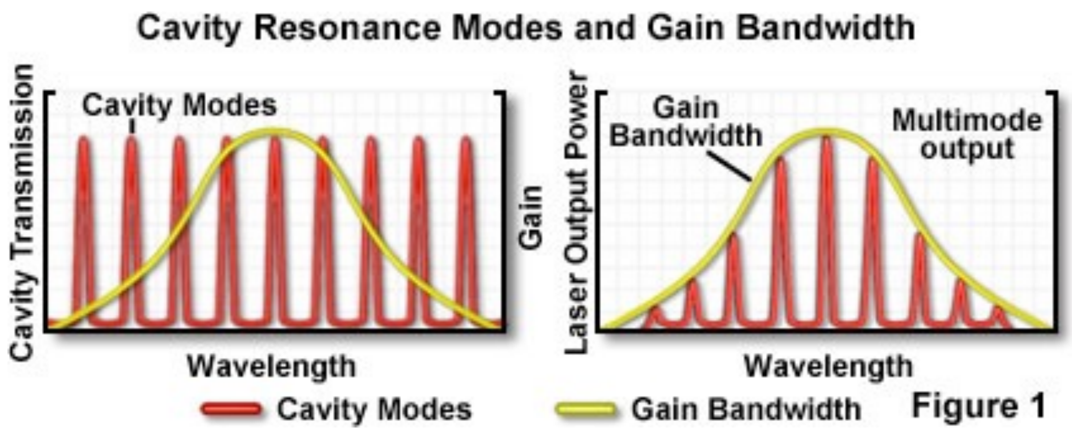
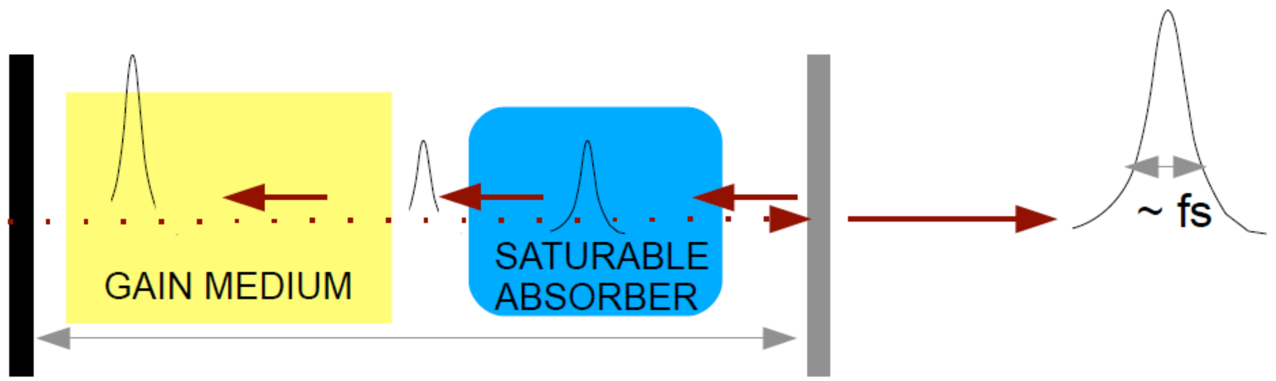


Ultrafast Multimode Laser

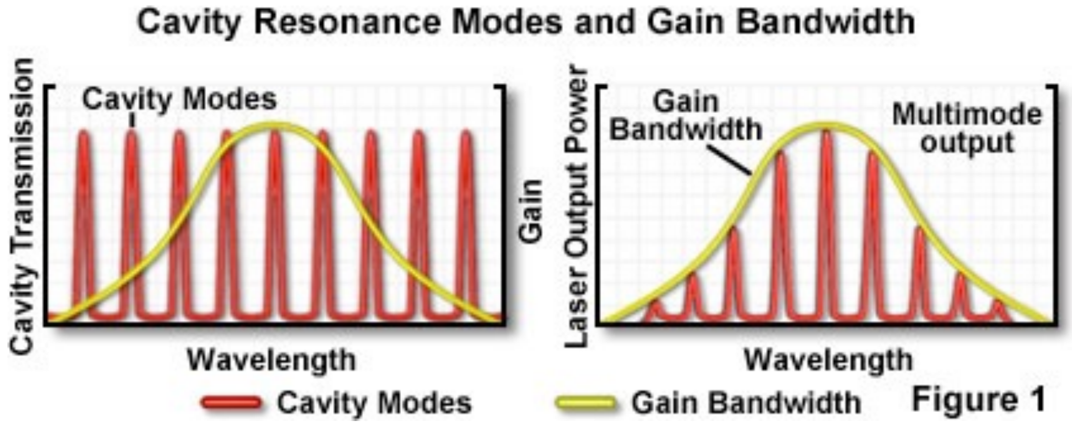


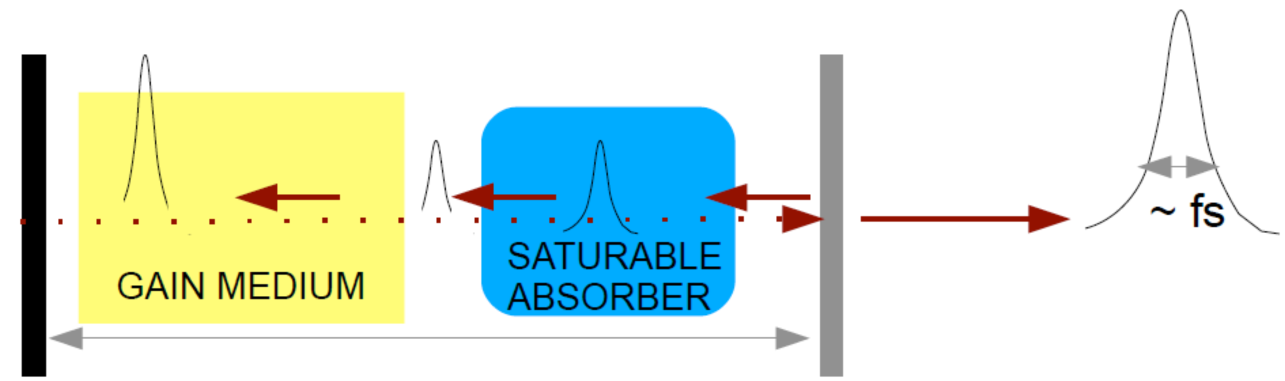


The saturable absorber induces self-starting synchronous oscillations of modes in the cavity: passive mode-locking -> fast pulses.

Related to a non-linear frequency matching condition occurring in the saturable absorber:

Ultrafast Multimode Laser





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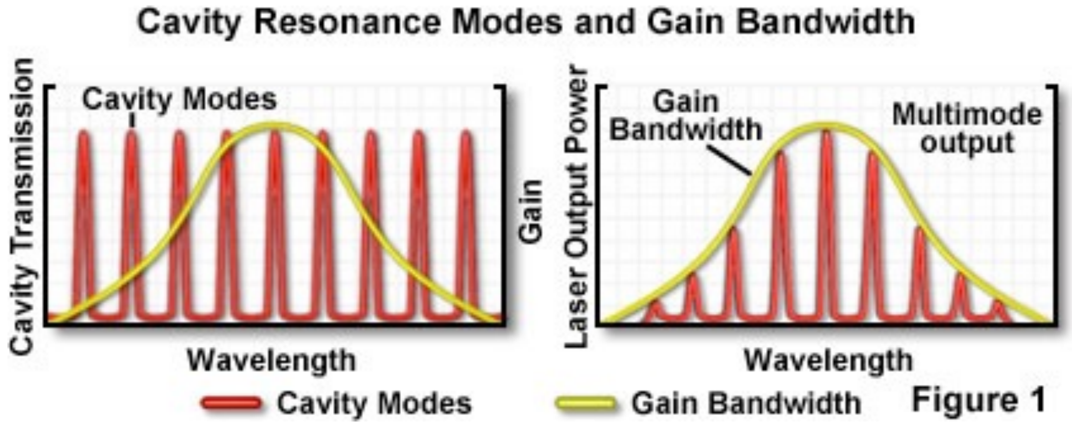
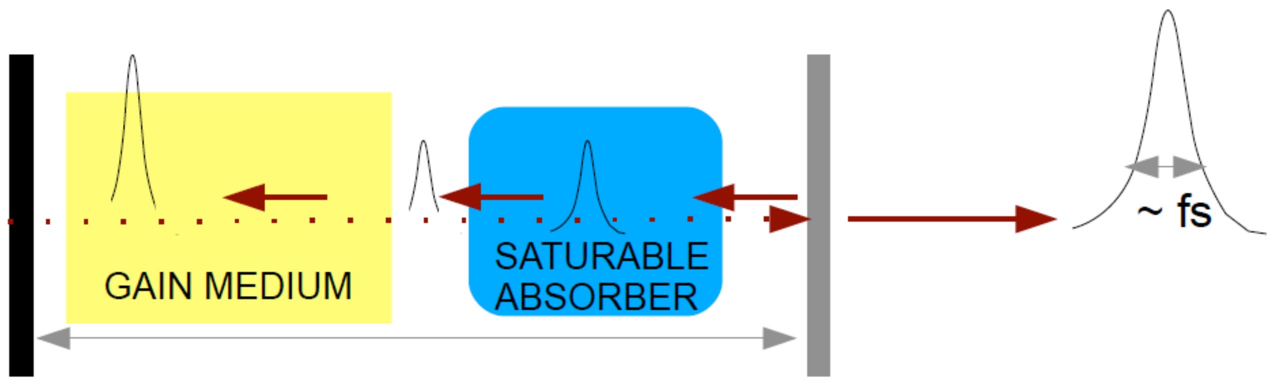
Ultrafast Multimode Laser





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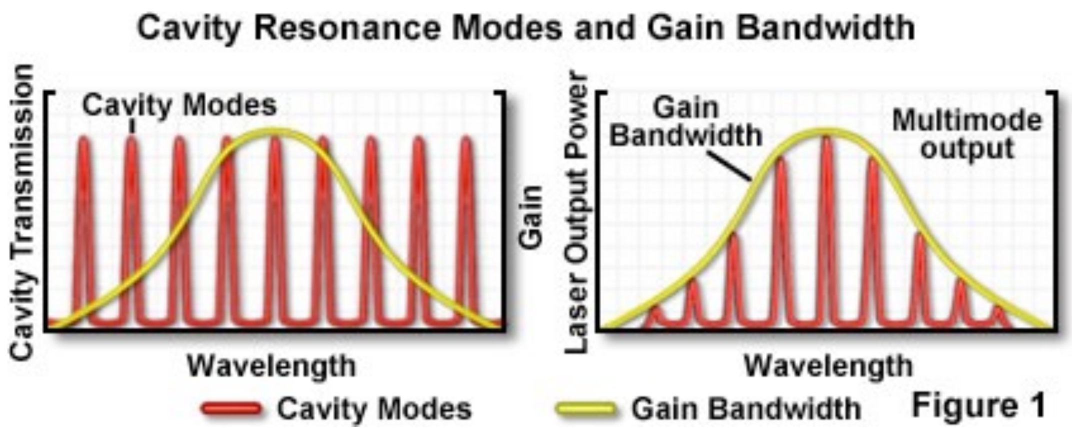
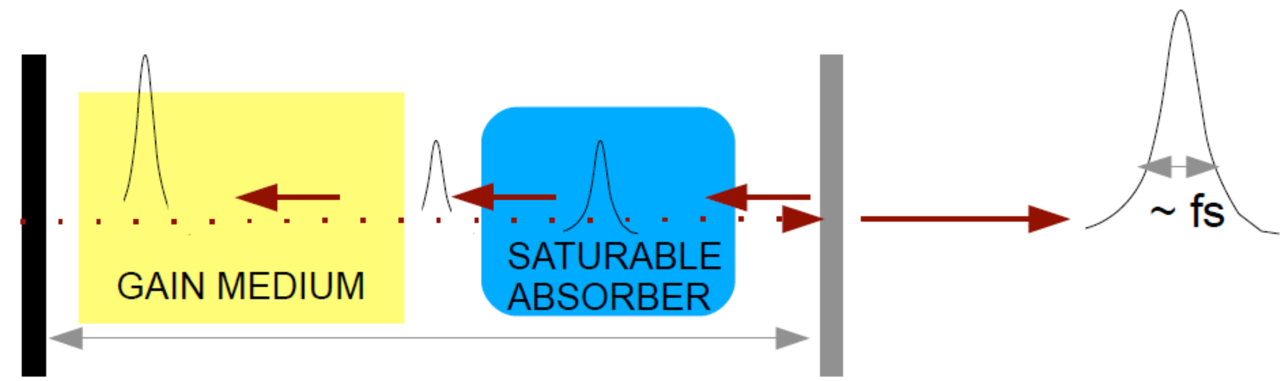
In the lasing regime, the phases of the amplified modes acquire a linear relationship to the frequencies:

$$\phi(\omega) = \phi_0 + \phi' \, \omega + O(\omega^2)$$

Ultrafast Multimode Laser





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Mode locking/Phase locking
takes place above the lasing optical power threshold.

HA Haus, *Mode-Locking of Lasers*, IEEE J. Quantum Electron., 2000

In the lasing regime, the phases of the amplified modes acquire a linear relationship to the frequencies:

$$\phi(\omega) = \phi_0 + \phi' \, \omega + O(\omega^2)$$

$$\phi_{n_1} - \phi_{n_2} + \phi_{n_3} - \phi_{n_4} = 0$$

ALL STANDARD MULTIMODE LASER THEORY SO FAR

Random laser

A Laser with nonresonant scatterer

Ambartsumyan, Basov, Kryukov, Lethokov (1966)

“Scatterer-mirror”, strong mode interaction due to scattering in different directions: there is coherent feedback but not on a narrow frequency interval -> “nonresonant”.

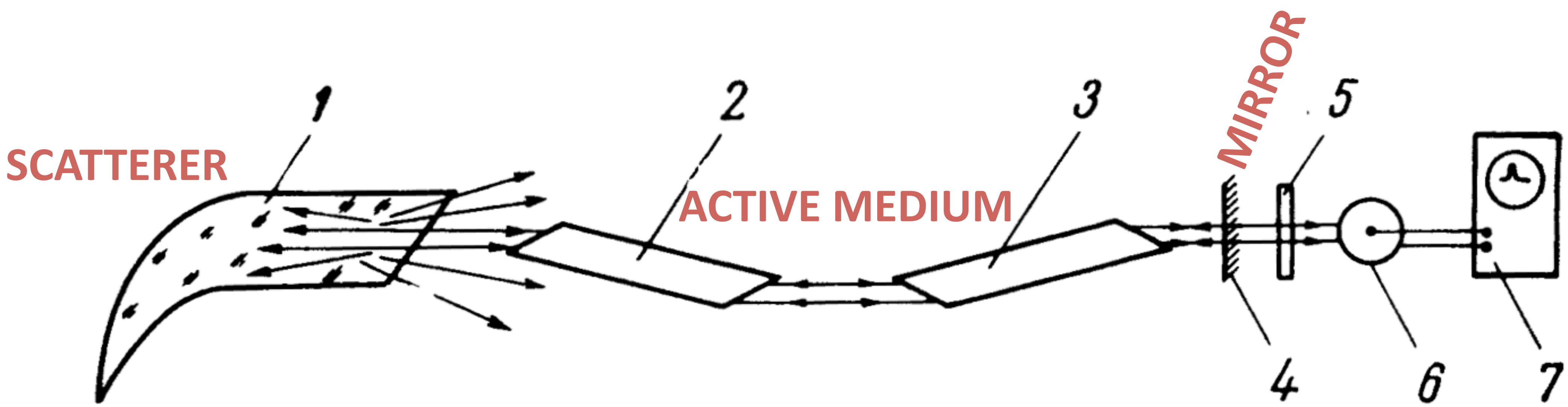


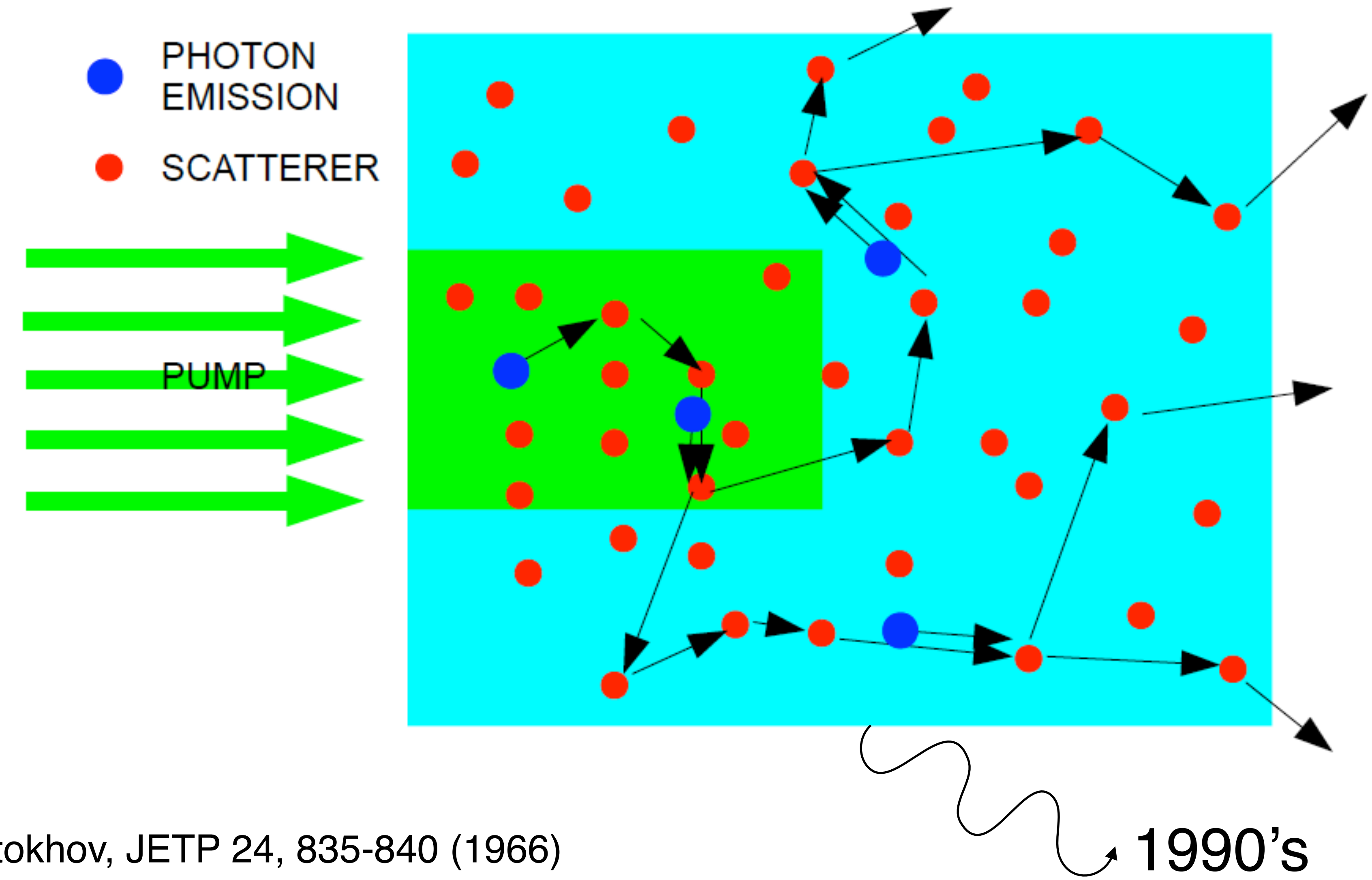
FIG. 1. Experimental setup—laser with scattering feedback.

Ambartsumyan, Basov, Kryukov, Lethokov,
IEEE Journal of Quantum Electronics 2, 442 - 446 (1966)
JETP 24, 481-485 (1967)

Random laser

Generation of Light by a Scattering Medium with Negative Resonance Absorption
Letokhov (1968)

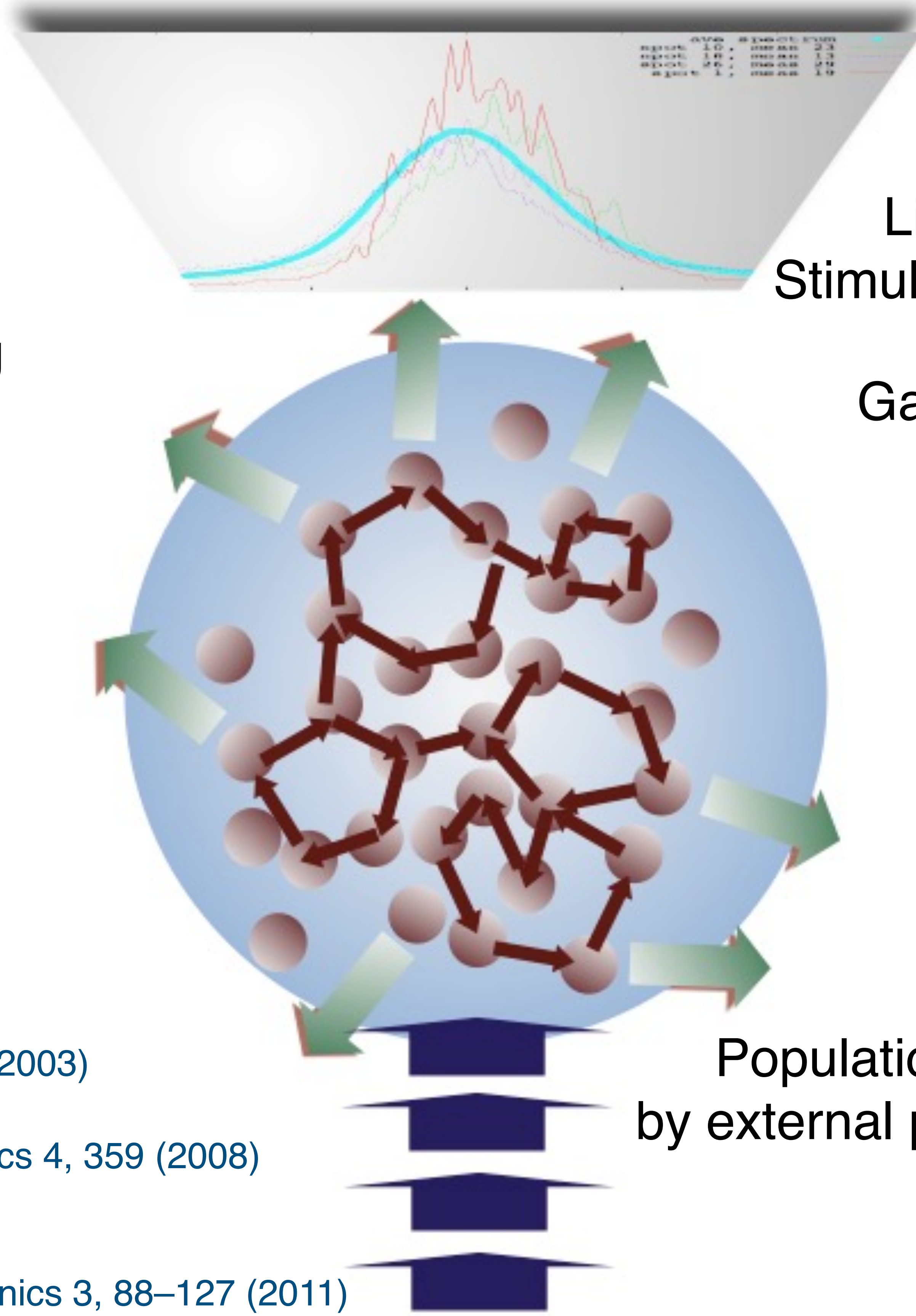
When photon path length is larger than amplification length: photon multiplication



"Photonic bomb"

Letokhov, JETP 24, 835-840 (1966)

Random laser



Light Amplification by Stimulated Emission Radiation

Multiple scattering of photons

Gain prevails over loss: standing modes

Population inversion by external power pumping

H. Cao, Waves in Random Media and Complex Media 13, R1–R39 (2003)

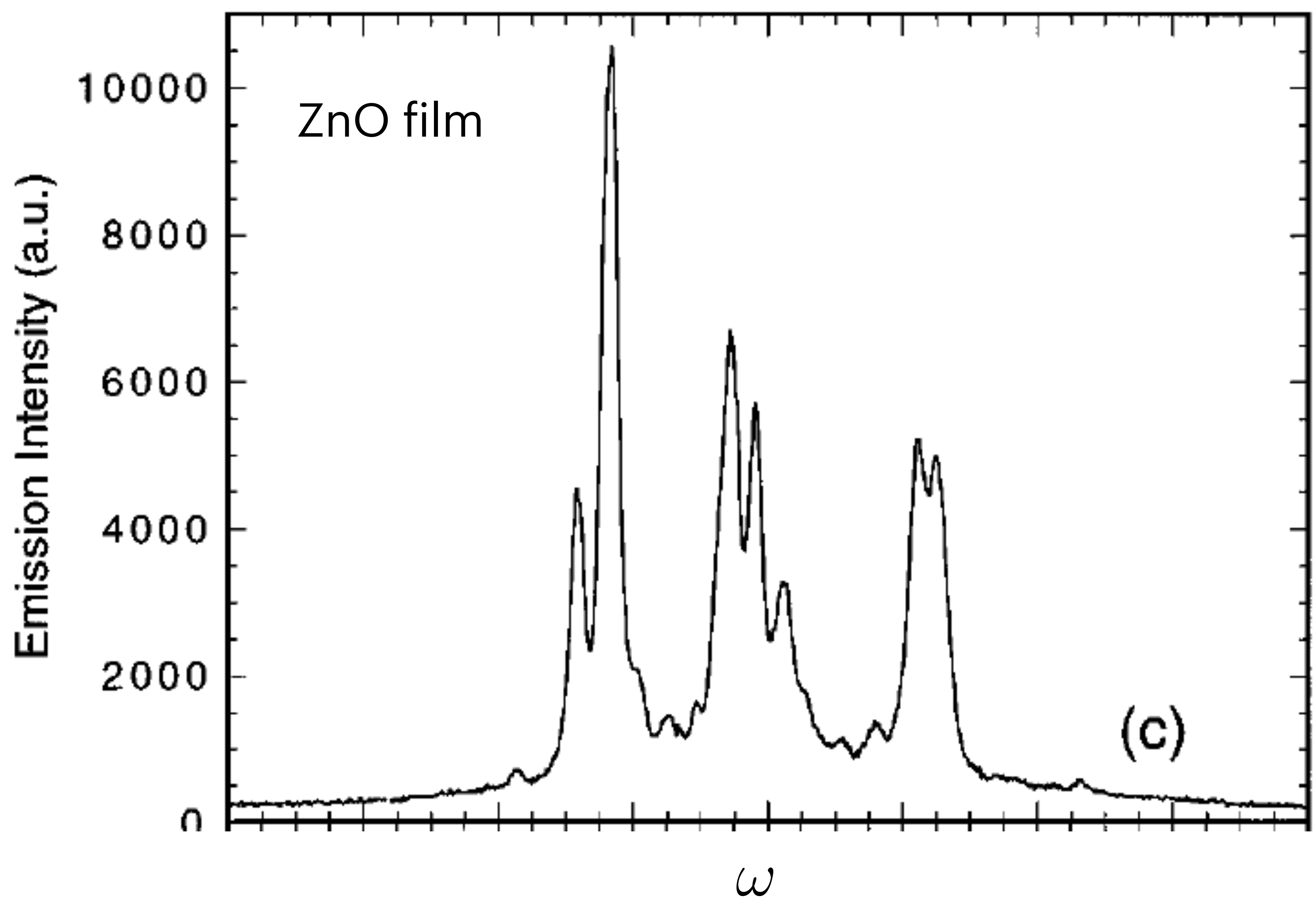
D. S. Wiersma, Nature Physics 4, 359 (2008)

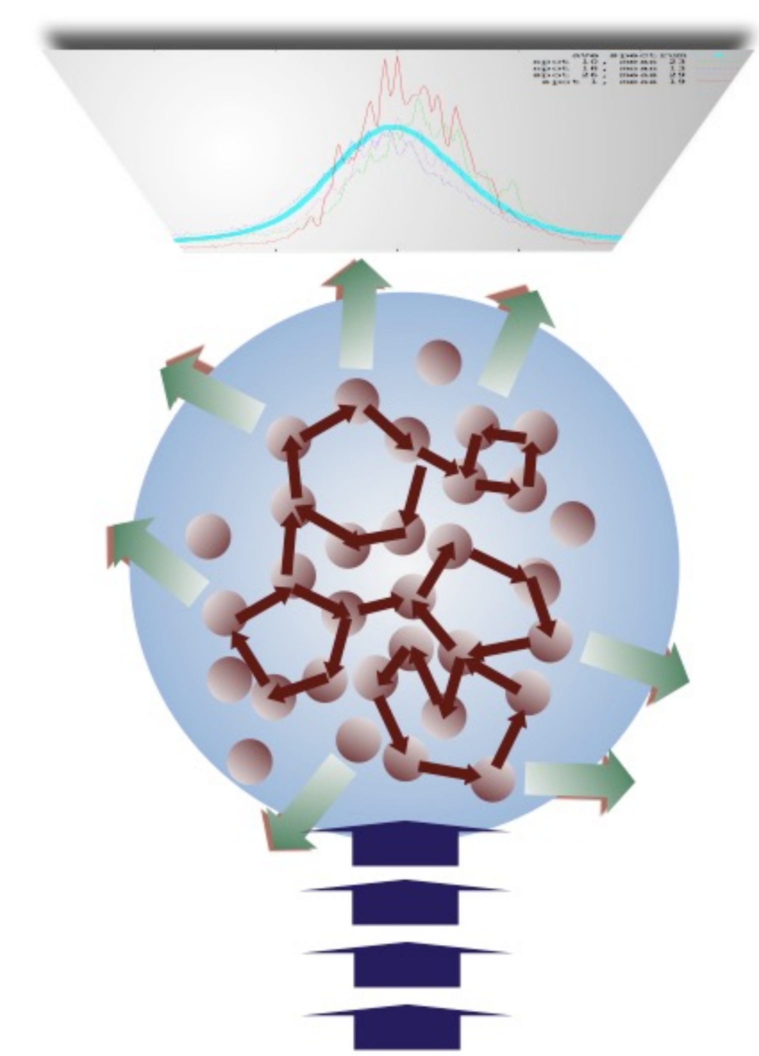
J. Andreasen et al.
Adv. Optics and Photonics 3, 88–127 (2011)

A. S. Gomes et al. Progress in Quantum Electronics 78, 100343 (2021)

Random laser emission spectrum at high pumping

Film of ZnO





H. Cao et al., Appl. Phys. Lett. 73, 3656 (1998)

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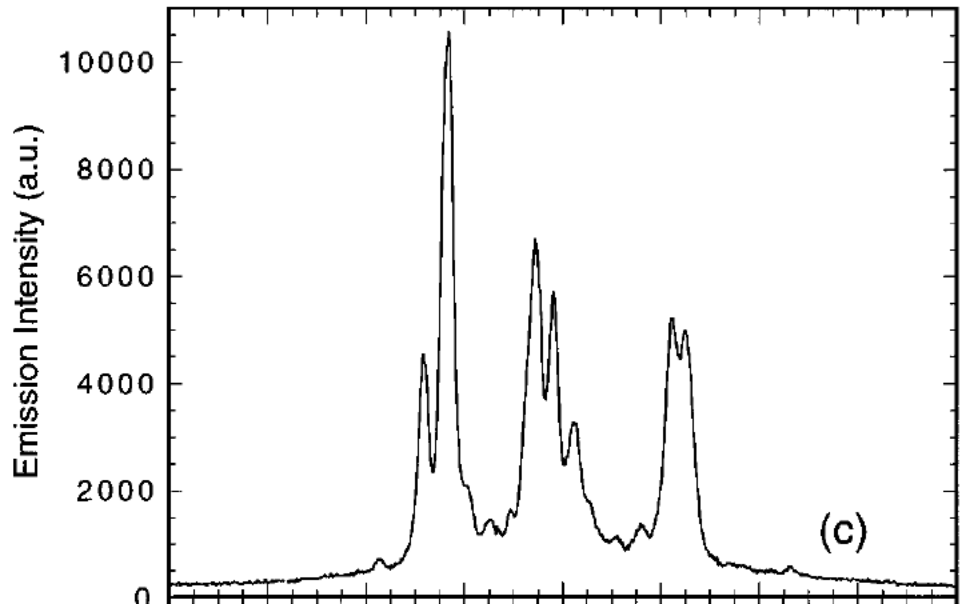
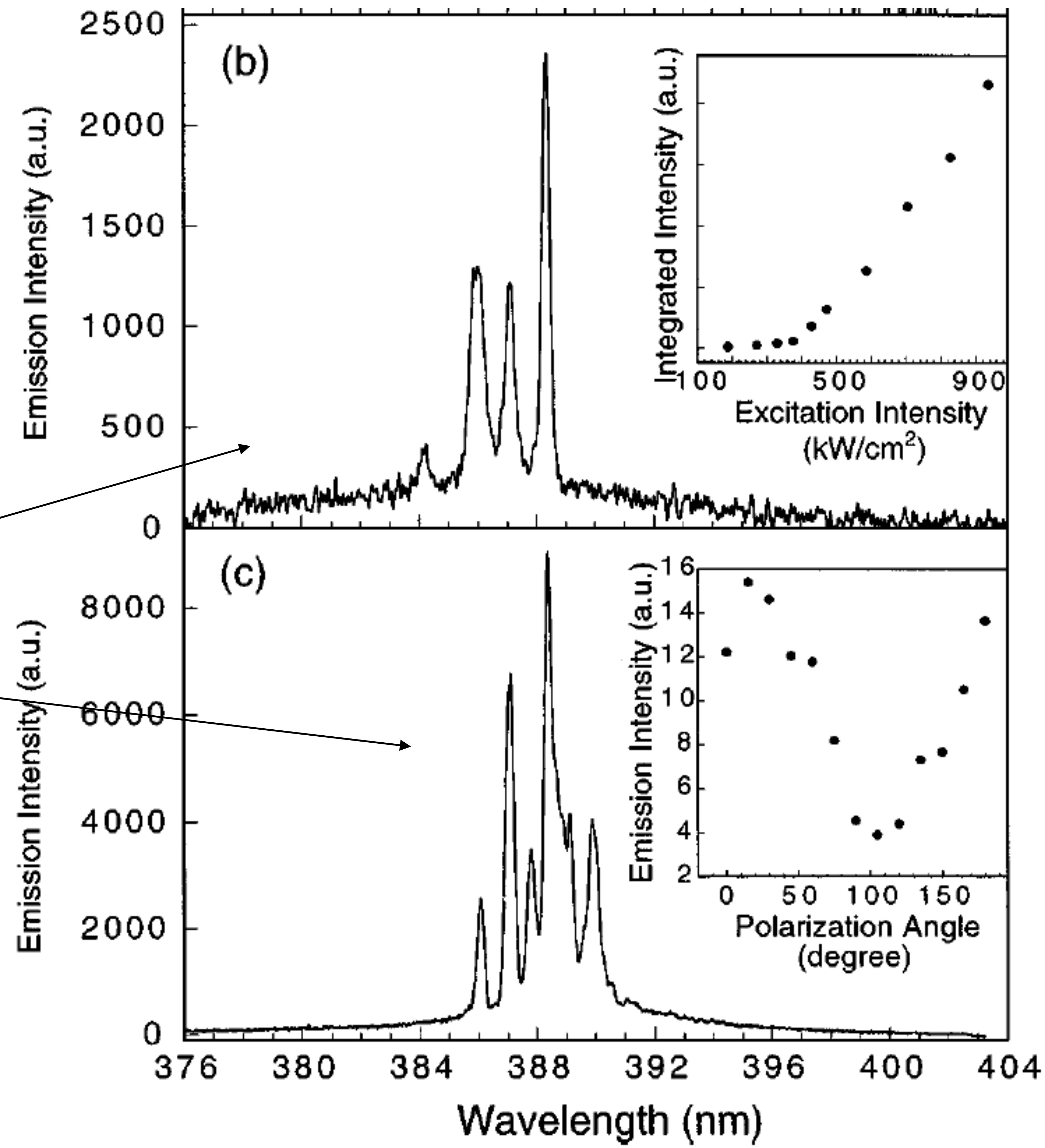




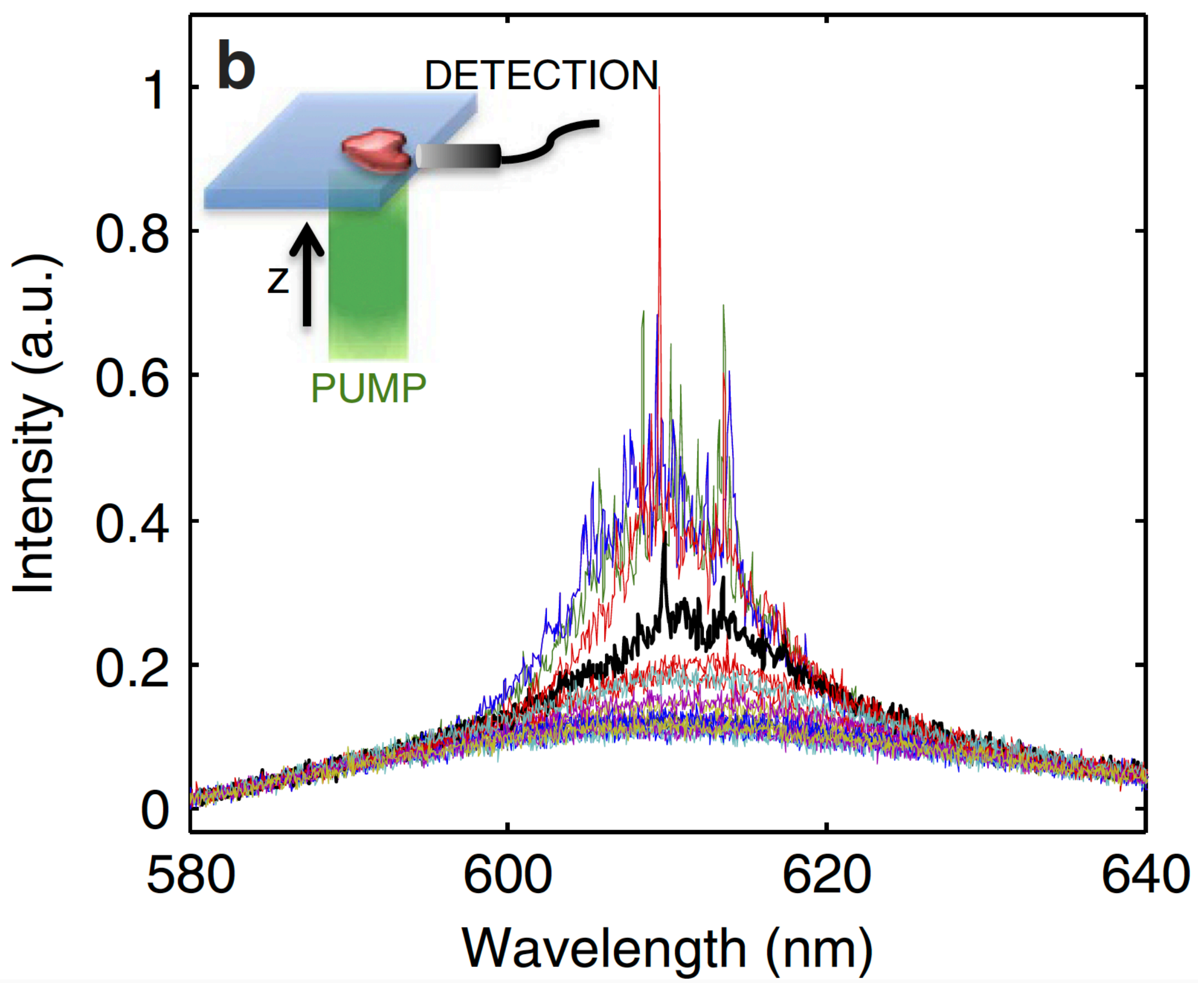
Changing the pumping power resonances move

ZnO film
H. Cao et al.,
Appl. Phys. Lett. 73, 3656 (1998)

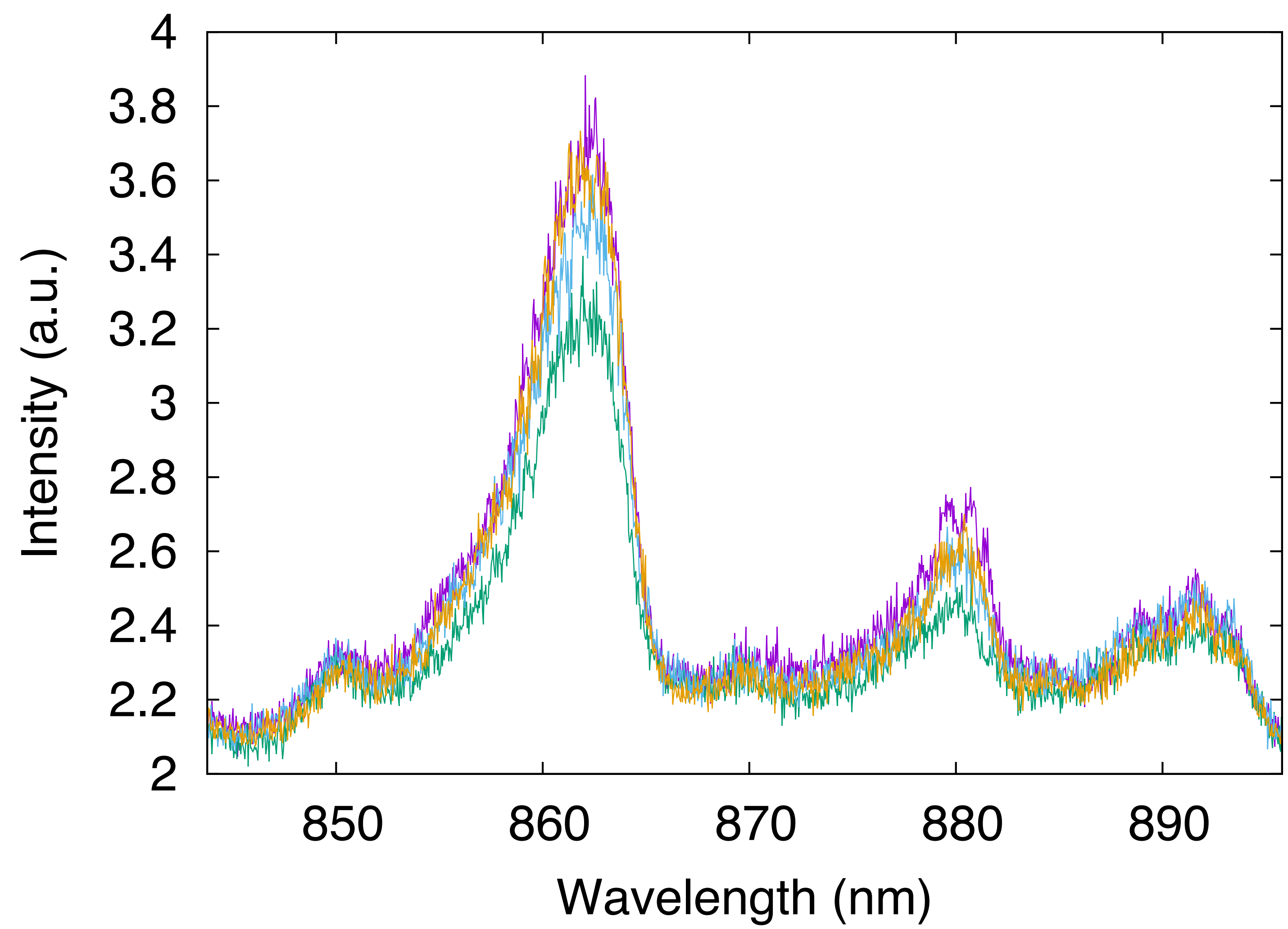
Random laser emission spectrum at high pumping

(In some random lasers) changing the pumping power the resonances move.
What happens at the emission after different shots at the same pumping power (aka at each realization of the same random laser)?

Sometimes resonances change a lot



Sometimes they just oscillate a bit



T5OCx
N Ghofraniha et al.,
Nat. Commun. 6, 6058 (2015)

GaAs powder
F Antenucci et al.,
PRL 126, 173901 (2021)

T5OCx = thienyl-S,S-dioxide quinquethiophene

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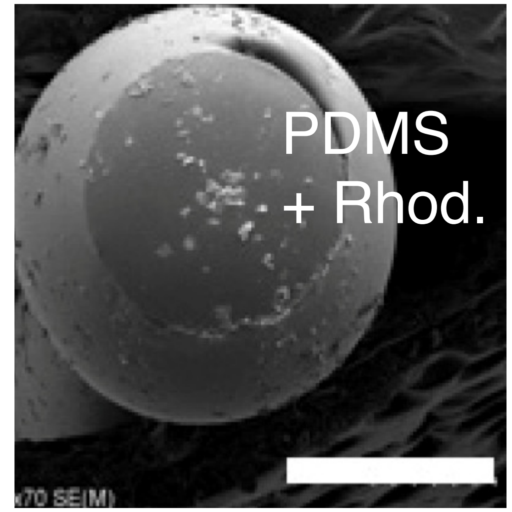
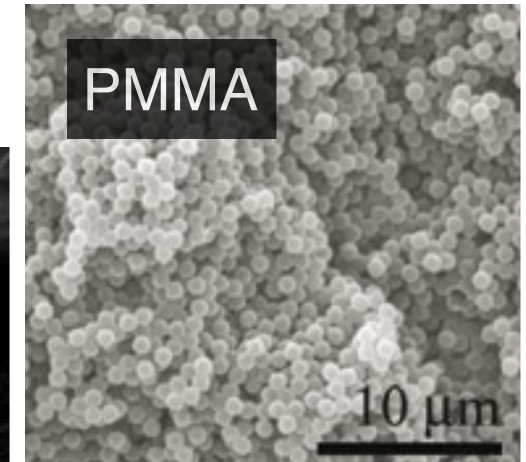
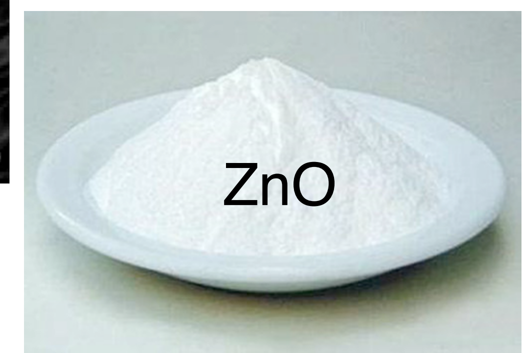
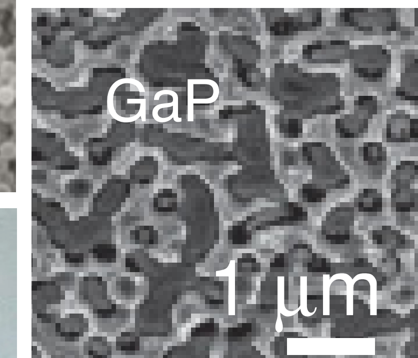
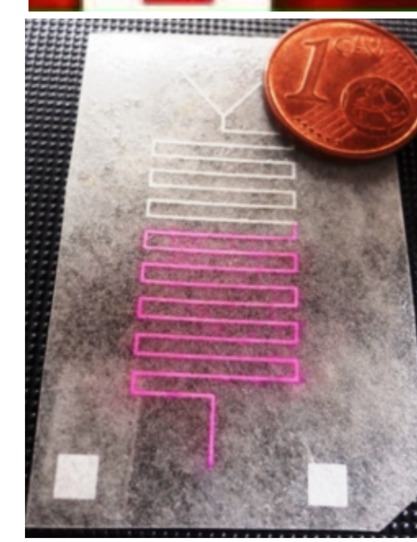
Many random laser materials











Many kinds of random matrices for random lasers:

- Photonics glass [Galisteo-Lo´pez et al. (2011)]
- Nanoparticles powders (TiO2, ZnO, GaN, GaAs)
- Porous media, [El-Dardiry et al., (2010)]
- Plasmonic waveguides [Zhai et al. (2011)]
- Quantum dots [Chen et al. (2011)]
- Disordered fibers [de Matos et al. (2007); Turitsyn et al. (2010)],
- Polymeric micro channels [Bhaktha et al. (2012)],
- Micro droplets [Tiwari et al. (2012)],
- Granular beads [Folli et al. (2013)],
- Paper [Viola et al. (2013); Ghofraniha et al. (2013)]
- Bio inspired materials [Wang et al. (2014)]
- Organic semiconductors*, used for LEDs and as gain in lasers, among which:
  - Organic conjugated polymers, PPV, MEH-PPV, DOO- PPV, PF [Moses (1992); Hide et al. (1996); Tessler et al. (1996); Holzer et al. (1996); Frolov et al. (1996); Polson and Vardeny (2005); Tulek and Vardeny (2010)];
  - Solutions of laser dyes (rhodamine 6G) with nanoparticles (TiO2, ZnO, GaN) [Cao et al. (2003); Wu et al. (2006); Polson and Vardeny (2005); Mujumdar et al. (2004)];
  - Organic-inorganic nanocomposites [Anglos et al. (2004)];
  - Organic nanofibers [Quochi et al. (2004, 2006); Andreev et al. (2006)];
  - Thiophene-based oligomers [Barbarella et al. (2005, 1999); Anni et al. (2004); Pisignano et al. (2002); Ghofraniha et al. (2013a)].

**Organic random lasers* in *Organic lasers*, Viola et al. 2018

Many more since 2018…..

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- Statistical physics approach to laser physics

Modeling multimode lasing with statistical mechanics

1) Electromagnetic field quantum dynamics can be mapped onto classical stochastic dynamics

Hackenbroich, Viviescas, Haken PRL, PRA 2003,

Antenucci, Crisanti, LL PRA 2015, Antenucci, Springer 2016.

2) Under stationarity conditions the system can be considered as if at equilibrium, coupled to an effective "thermal" *reservoire*

Ordered lasers: Gordon, Fisher PRL 2002, Opt. Commun. 2003;
Gat, Gordon, Fisher, PRE 2004; Weill et al PRL 2005;
Antenucci, Ibanez Berganza, LL PRA, PRB 2015; Marruzzo, LL PRB 2015.

Random lasers (quenched homogeneous amplitudes — phase only):
Angelani et al. PRL, PRB 2006; LL et al. PRL 2009; Conti, LL PRB 2011;

Random lasers: Antenucci, Conti, Crisanti, LL PRL 2015; Antenucci, Crisanti, LL SciRep 2015;
Marruzzo, LL PRB 2016; Antenucci et al. Phil. Mag. 2016; **Antenucci, Springer 2016.**

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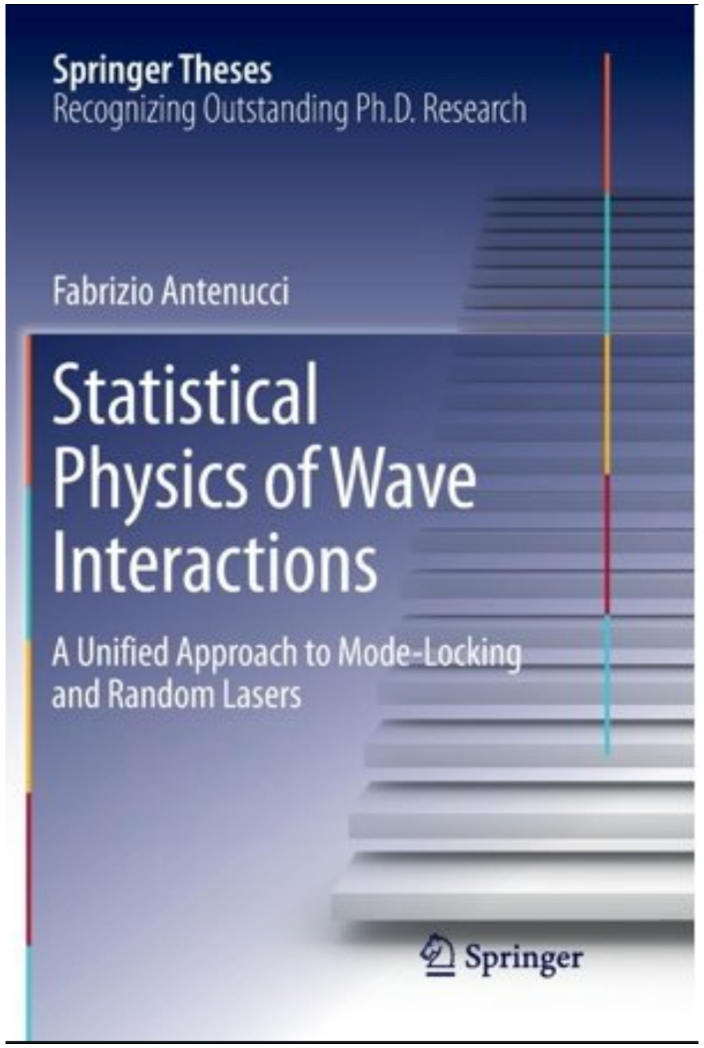
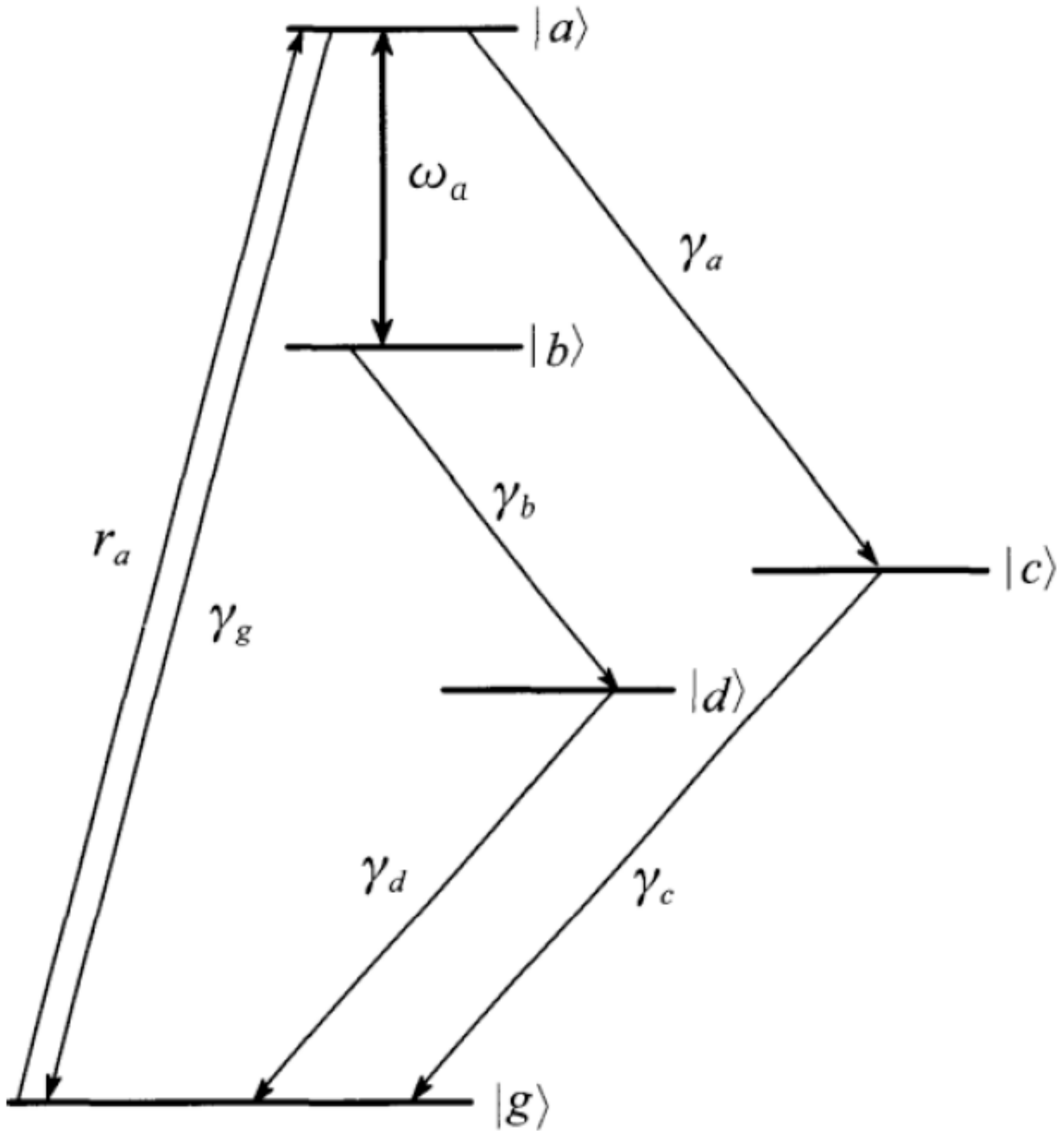
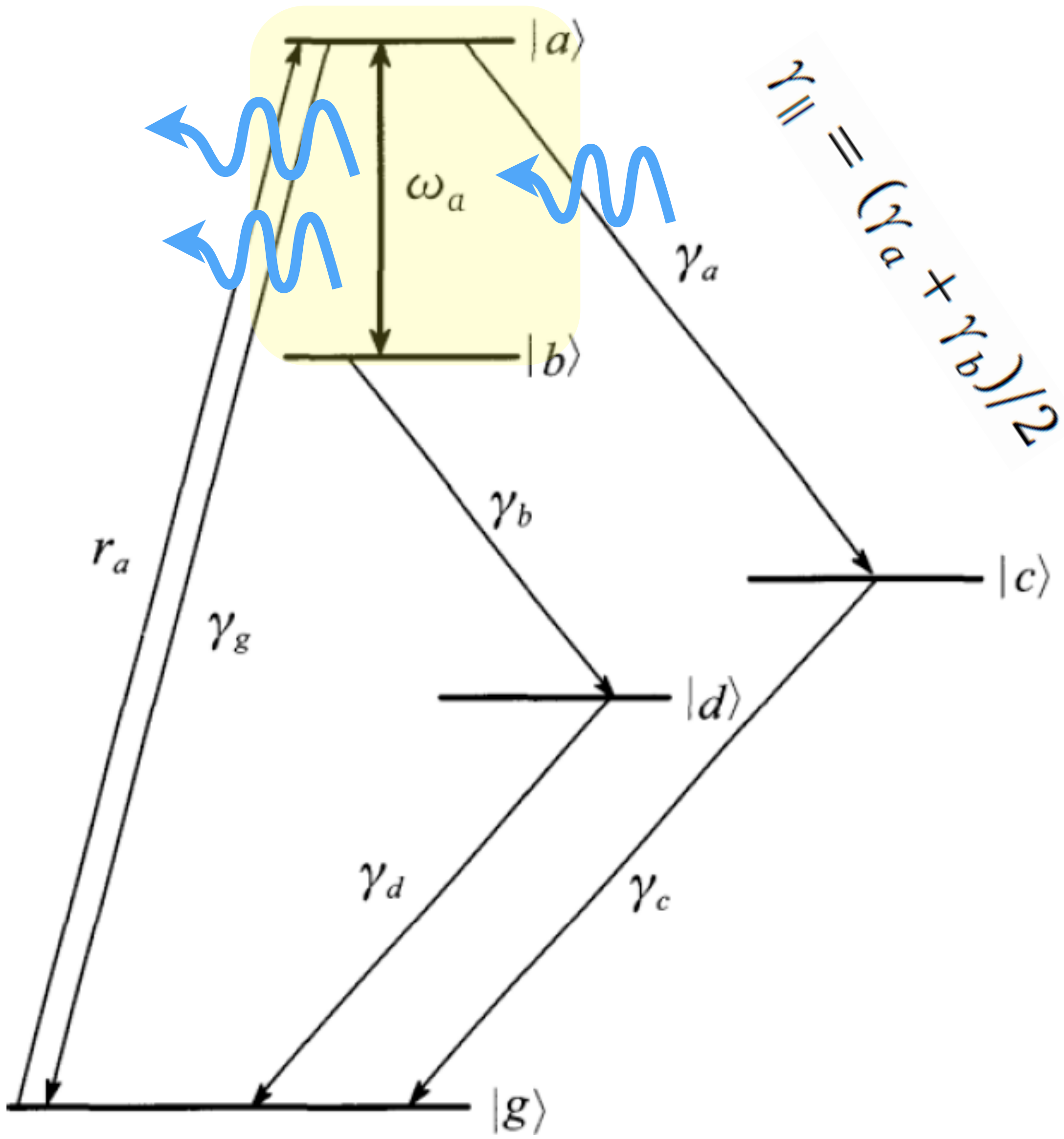


- Statistical physics approach to laser physics



• Statistical physics approach to laser physics



RADIATION

$$a_\lambda,\ a_\lambda^\dagger$$

creation, annihilation operators of the e.m. field

MATTER

$$\sigma_-(\boldsymbol{r}) \equiv |b\rangle\langle a|$$

lowering

$$\sigma_+(\boldsymbol{r}) \equiv |a\rangle\langle b|$$

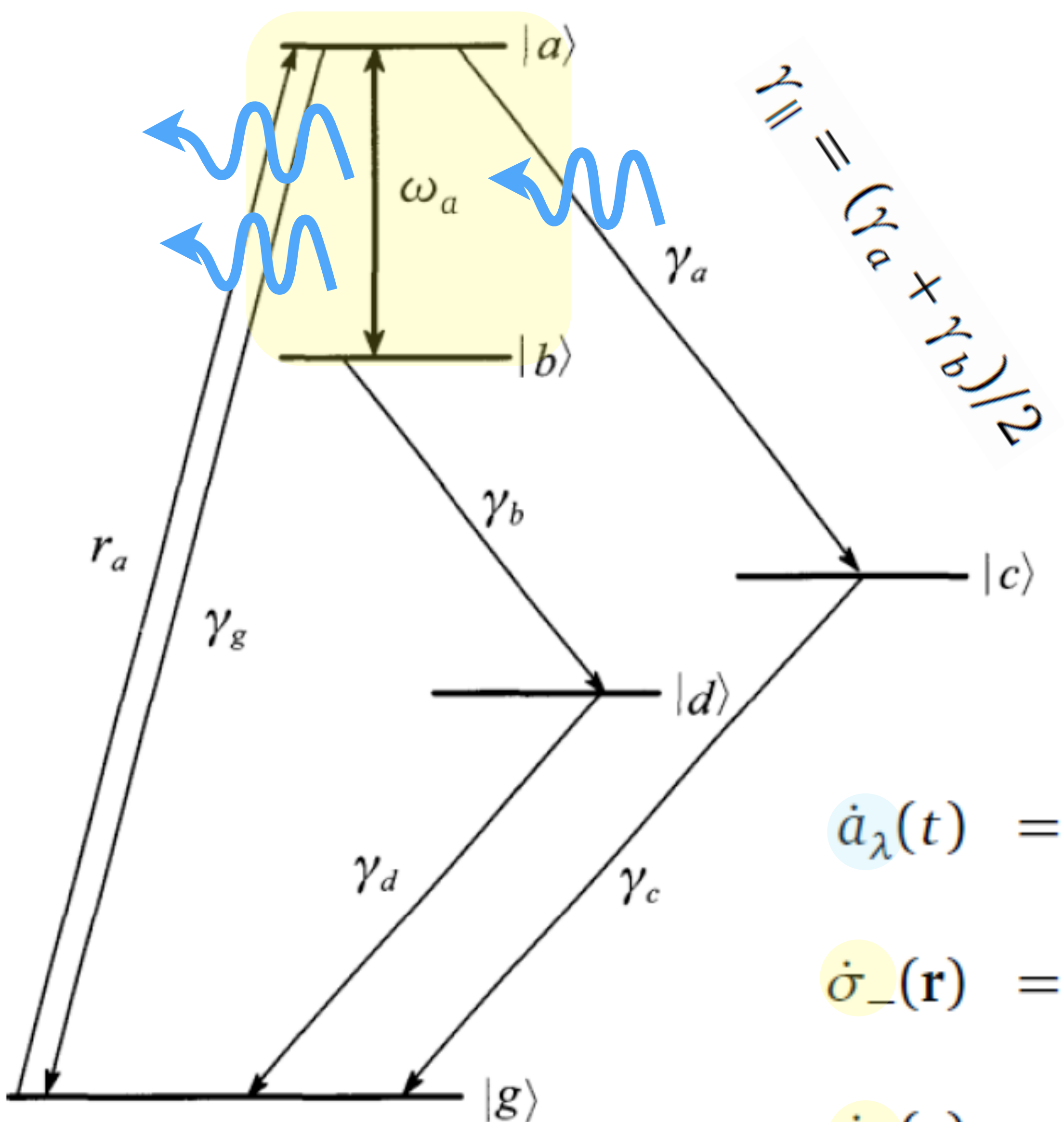
raising

$$\sigma_z(\boldsymbol{r}) \equiv |a\rangle\langle a| - |b\rangle\langle b|$$

population inversion

• Statistical physics approach to laser physics

Jaynes-Cummings quantum stochastic dynamics for light-matter interaction



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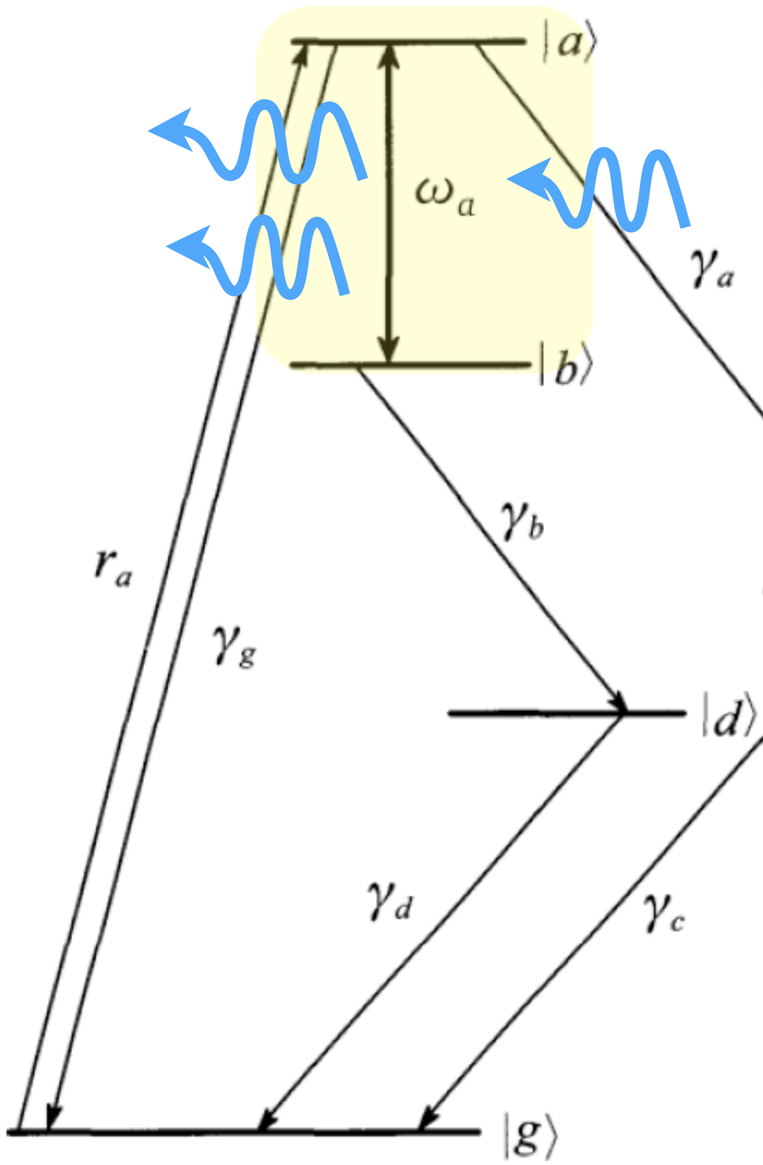
population inversion

$$\begin{aligned}
\dot a_\lambda(t) &= -i\omega_\lambda a_\lambda(t) + \int d\mathbf{r}\, g^*_\lambda(\mathbf{r})\sigma_-(\mathbf{r}) - \sum_{\lambda'} \Gamma_{\lambda\lambda'} a_{\lambda'}(t) + F_\lambda(t)\\
\dot\sigma_-(\mathbf{r}) &= -(\gamma_\perp + i\omega_a)\sigma_-(\mathbf{r}) + 2\sum_\lambda g_\lambda(\mathbf{r})\sigma_z(\mathbf{r}) a_\lambda + F_-(\mathbf{r})\\
\dot\sigma_z(\mathbf{r}) &= \gamma_\parallel (S\rho(\mathbf{r}) - \sigma_z(\mathbf{r})) - \sum_\lambda \left[g^*_\lambda(\mathbf{r}) a^\dagger_\lambda \sigma_-(\mathbf{r}) + h.c. \right] + F_z(\mathbf{r})
\end{aligned}$$

• Statistical physics approach to laser physics

Jaynes-Cummings quantum stochastic dynamics for light-matter interaction



$\gamma_{\parallel} = (\gamma_a + \gamma_b)/2$

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population inversion

$$\dot{a}_\lambda(t) = -i\omega_\lambda a_\lambda(t) + \int d\mathbf{r}\, g^*_\lambda(\mathbf{r})\sigma_-(\mathbf{r}) - \sum_{\lambda'} \Gamma_{\lambda\lambda'} a_{\lambda'}(t) + F_\lambda(t)$$

$$\dot{\sigma}_-(\mathbf{r}) = -(\gamma_\perp + i\omega_a)\sigma_-(\mathbf{r}) + 2\sum_\lambda g_\lambda(\mathbf{r})\sigma_z(\mathbf{r}) a_\lambda + F_-(\mathbf{r})$$

$$\dot{\sigma}_z(\mathbf{r}) = \gamma_{\parallel}(S\rho(\mathbf{r}) - \sigma_z(\mathbf{r})) - \sum_\lambda \left[g^*_\lambda(\mathbf{r}) a^\dagger_\lambda \sigma_-(\mathbf{r}) + h.c.\right] + F_z(\mathbf{r})$$

Downgrading operators to complex numbers we obtain a classical description of the mode amplitudes stochastic dynamics

- Statistical physics approach to laser physics

Random laser stochastic differential equations for the complex amplitudes (the *phasors*)

$$\dot{a}_n(t) = \mathcal{F}_n[\{a\}|\{J\}] + \eta_n(t) \qquad n = 1, \ldots, N$$

T is related to spontaneous emission, i.e., to the real temperature of the system

Potential solution to the Fokker-Planck equation

Boltzmann-Gibbs like distribution of the amplitudes' configurations, at some effective temperature

Ordered lasers:
Gordon, Fisher PRL 89, 103901 (2002); Opt. Commun. 223, 151 (2003); Gat, Gordon, Fisher, PRE 70, 046108 (2004);
Random lasers, quenched homogeneous amplitude approx:
L Angelani et al. PRL 96, 065702 (2006); PRB 74, 104207 (2006), LL et al. PRL 102, 083901 (2009), Conti, LL PRB 83, 134204 (2011)
Random lasers, phasors:
F Antenucci, C Conti, A Crisanti, LL, PRL 114, 043901 (2015); F Antenucci, A Crisanti, LL, PRA 91, 053816 (2015), F Antenucci et al Phil. Mag. 96, 704-731 (2016), F. Antenucci, Springer 2016

- Statistical physics approach to laser physics

Random laser stochastic differential equations for the complex amplitudes (the *phasors*)

$$\dot{a}_n(t) = \mathcal{F}_n[\{a\}|\{J\}] + \eta_n(t) \qquad n = 1, \ldots, N$$

$$\langle \eta_n(t) \rangle = 0; \qquad \langle \eta_n(t)\eta_m(s) \rangle = 2T\delta(t-s)\gamma_{nm} \simeq 2T\delta(t-s)\delta_{nm}$$

T is related to spontaneous emission, i.e., to the real temperature of the system

Potential solution to the Fokker-Planck equation

$$\mathcal{F}_n[\{a\}|\{J\}] \sim -\frac{\partial}{\partial a_n^*}\, \mathcal{H}[\{a\}|\{J\}]$$

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- Statistical physics approach to laser physics

What are the phasors $\{a\}$, variables/degrees of freedom of our theory?

They are the complex amplitudes in the slow amplitude e.m. field expansion in normal modes

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_k \underbrace{a_k(t)}_{\text{slow amplitude complex coefficient}} \underbrace{\boldsymbol{E}_k(\boldsymbol{r})}_{\text{normal mode}} \, e^{\imath \overbrace{\omega_k}^{\text{mode frequency}} t} + \text{c.c.}$$

- Statistical physics approach to laser physics

Laser stationary regime and equilibrium statistical mechanics

Lasers are not at equilibrium: energy is pumped to maintain the population inversion and in open cavities energy is lost by radiation.

Yet lasers are stable and stationary
Because of **gain saturation**

- Statistical physics approach to laser physics

Laser stationary regime and equilibrium statistical mechanics

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gain is the amplification factor of the output to input power $\longleftrightarrow$ atomic level populations $\longleftrightarrow$ optical mode intensities

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gain is the amplification factor of the output to input power $\longleftrightarrow$ atomic level populations $\longleftrightarrow$ optical mode intensities

Yet lasers are stable and stationary
Because of **gain saturation**

as the energy shared by the modes (number of photons emitted) increases the gain is depleted. The energy of modes consequently decreases, then gain increases...

$$g(\mathcal{E}) = \frac{g_0}{1+\frac{\mathcal{E}}{E_{\rm sat}}}$$

$$\simeq g_0 - \frac{g_0 \mathcal{E}}{E_{\rm sat}}$$

$$\mathcal{E} \ll E_{\rm sat}$$

weak saturation

$$\boxed{\mathcal{E}(t)} = \int_t^{t+\mathcal{T}_{\rm fast}} d\tau |a(t,\tau)|^2 = \int_t^{t+\mathcal{T}_{\rm fast}} d\tau \sum_{n,m} e^{\imath(\omega_n-\omega_m)\tau} a(t,\omega_n)a^*(t,\omega_m)$$

$$\propto \sum_{n,m} \delta(\omega_n-\omega_m) a(t,\omega_n)a^*(t,\omega_m) = \boxed{\sum_n |a(t,\omega_n)|^2}$$

$\mathcal{T}_{\rm fast}$ roundtrip time, stochastic resonator period

$t \gg \mathcal{T}_{\rm fast}$

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- Statistical physics approach to laser physics

Laser stationary regime and equilibrium statistical mechanics

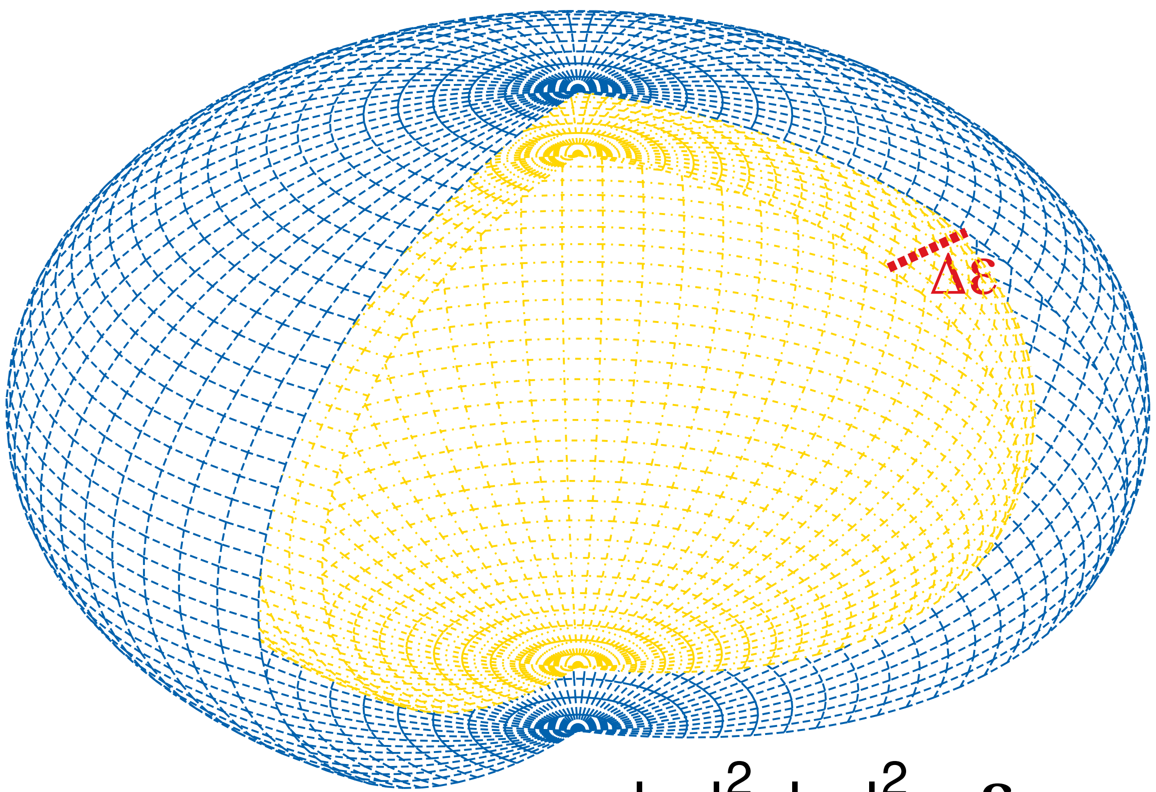
Lasers are not at equilibrium: energy is pumped to maintain the population inversion and in open cavities energy is lost by radiation.

gain is the amplification factor of the output to input power ⟷ atomic level populations ⟷ optical mode intensities

Yet lasers are stable and stationary
Because of **gain saturation**

$$g(\mathcal{E}) = \frac{g_0}{1+\frac{\mathcal{E}}{E_{\rm sat}}}$$

$$\mathcal{E} = \sum_n |a_n(\omega_n)|^2$$



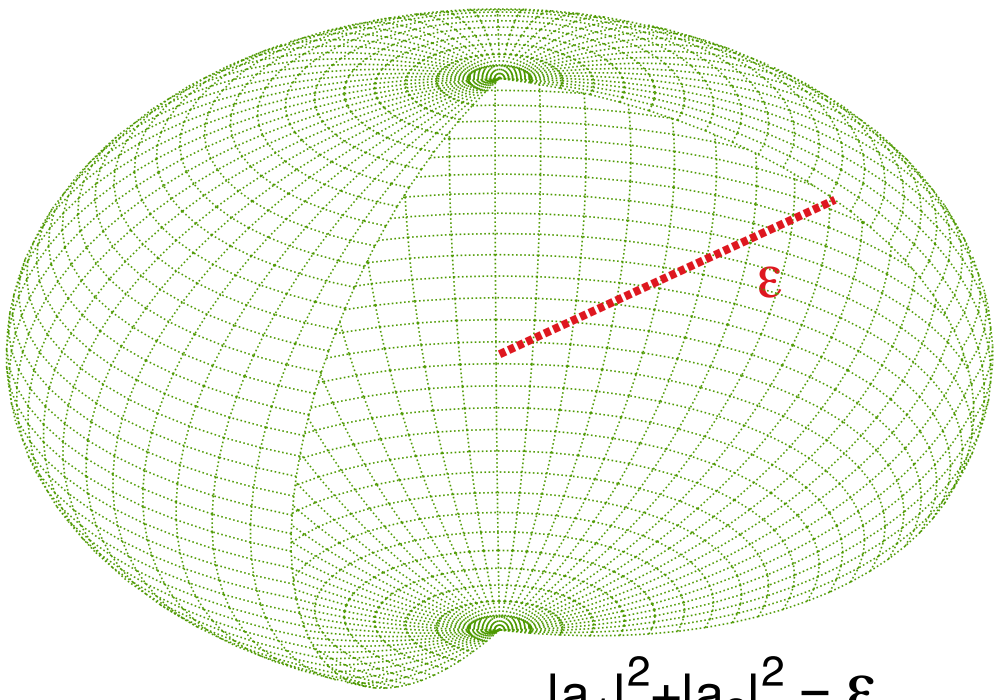
- Statistical physics approach to laser physics

Laser stationary regime and equilibrium statistical mechanics: encoding gain saturation

$$\mathcal{E} = \sum_n |a_n(\omega_n)|^2 = N\epsilon$$

As the total optical power of the amplified modes is kept strictly constant
— quenched dynamics —

$$\dot{\mathcal{E}} = 0$$



Gain saturation yields a global constraint on the overall intensity of all phasors

- Statistical physics approach to laser physics

Laser stationary regime and equilibrium statistical mechanics: encoding gain saturation

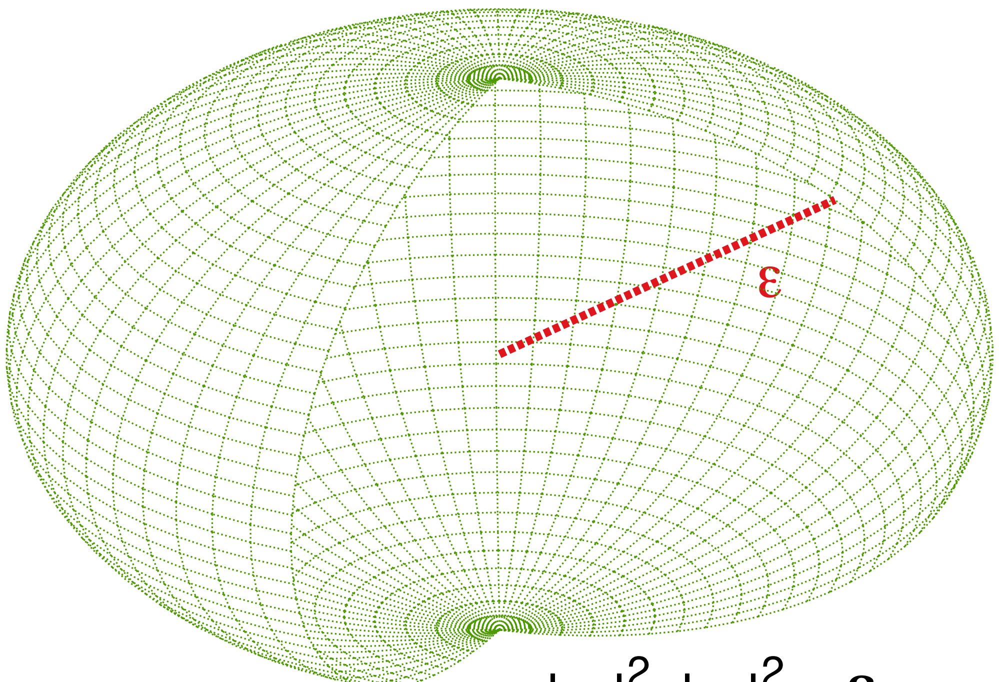
$$\mathcal{E} = \sum_n |a_n(\omega_n)|^2 = N\epsilon$$

As the total optical power of the amplified modes is kept strictly constant — quenched dynamics —

$$\dot{\mathcal{E}} = 0$$

the related stationary system can be considered as if at equilibrium with an effective "heat-bath" at the "photonic" temperature

$$\text{“}T_{\rm laser}\text{”} = \frac{T}{\epsilon^2}$$



$|a_1|^2+|a_2|^2 = \varepsilon$

We will also use the PUMPING RATE parameter $\mathcal{P} = \epsilon\sqrt{\beta J_0}$

Gain saturation yields a global constraint on the overall intensity of all phasors

• Statistical physics approach to laser physics

Random laser stochastic differential equations for the complex amplitudes (the *phasors*)

$$\dot{a}_n(t) = \mathcal{F}_n[\{a\}|\{J\}] + \eta_n(t) \qquad n = 1, \ldots, N$$

$$\langle \eta_n(t) \rangle = 0; \qquad \langle \eta_n(t)\eta_m(s) \rangle = 2T\delta(t-s)\gamma_{nm} \simeq 2T\delta(t-s)\delta_{nm}$$

T is related to spontaneous emission, i.e., to the real temperature of the system

Gain saturation yields a global constraint on the overall intensity of all phasors

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2 = \text{const}$$

Potential solution to the Fokker-Planck equation

$$\mathcal{F}_n[\{a\}|\{J\}] \sim -\frac{\partial}{\partial a_n^*} \mathcal{H}[\{a\}|\{J\}]$$

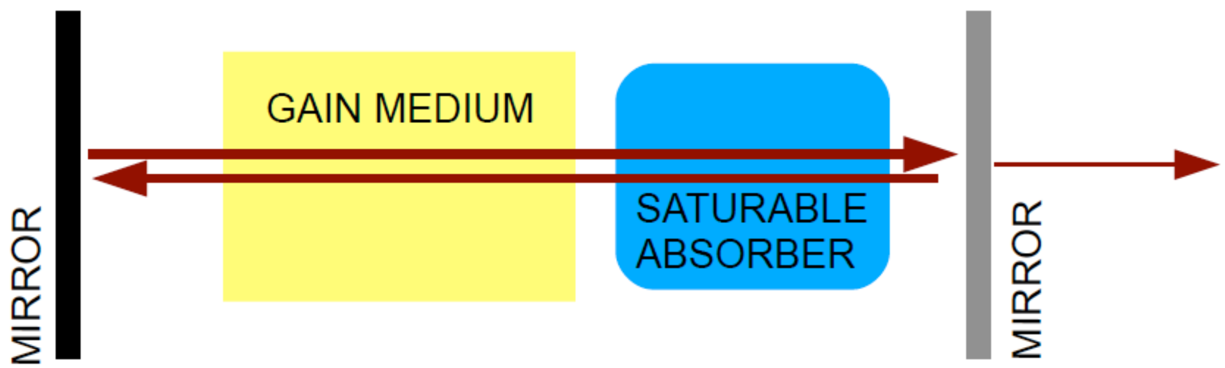
Boltzmann-Gibbs like distribution of the amplitudes' configurations, with

$$\text{“}T_{\text{laser}}\text{”} = \frac{T}{\epsilon^2}$$

Ordered lasers: Gordon, Fisher PRL 89, 103901 (2002); Opt. Commun. 223, 151 (2003); Gat, Gordon, Fisher, PRE 70, 046108 (2004);
Random lasers, quenched homogeneous amplitude approx: L Angelani et al. PRL 96, 065702 (2006); PRB 74, 104207 (2006), LL et al. PRL 102, 083901 (2009), Conti, LL PRB 83, 134204 (2011)
Random lasers, phasors:
F Antenucci, C Conti, A Crisanti, LL, PRL 114, 043901 (2015); F Antenucci, A Crisanti, LL, PRA 91, 053816 (2015), F Antenucci et al Phil. Mag. 96, 704-731 (2016), F. Antenucci, Springer 2016

- Theory for ultrafast mode-locked multimode lasers (order, closed cavity)

Link to Standard Mode-Locking



The equation for the completely closed and ordered limit is known:
Haus standard Mode-Locked laser master equation (1984)

$$\dot{a}_n = (g_m - \ell_m + \imath D_m) a_n + (\gamma - \imath \delta) \sum_{\omega_j - \omega_k + \omega_l = \omega_n} a_j a_k^* a_l + \eta_n$$

HA Haus, *Waves and Fields in Optoelectonics, 1984*
HA Haus, *Mode-Locking of Lasers*, IEEE J. Quantum Electron., 2000
Gordon & Fischer, PRL 2002; Opt. Commun. 2003; Gat, Gordon, Fisher PRE 2004

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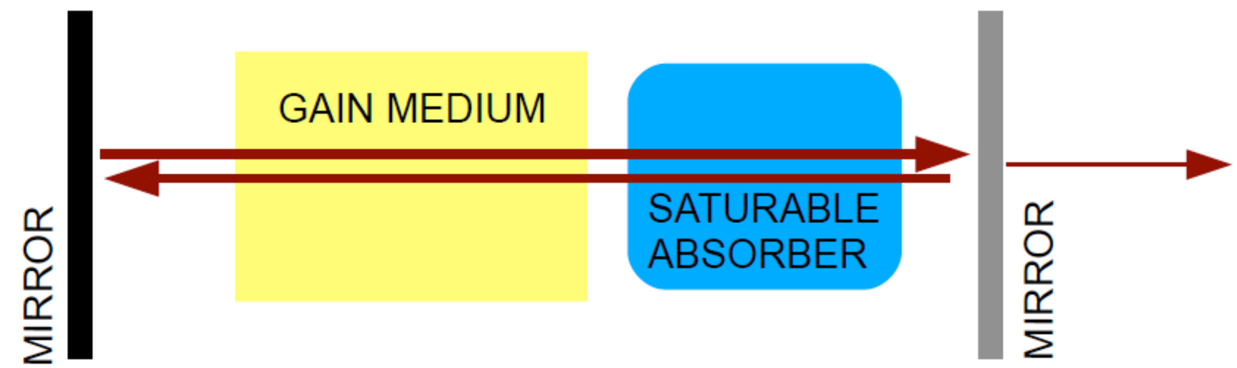

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SAM: SELF-AMPLITUDE MODULATION COEFFICIENT OF THE SATURABLE ABSORBER

$$\dot a_n = \big(\underbrace{g_m}_{\text{GAIN}} - \overbrace{\ell_m}^{\text{LOSS}} + \underbrace{\imath D_m}_{\text{GROUP VELOCITY DISPERSION}}\big) a_n + \big(\overbrace{\gamma}^{\text{SAM}} - \underbrace{\imath\delta}_{\text{KERR LENS}}\big) \sum_{\omega_j - \omega_k + \omega_l = \omega_n} a_j a_k^* a_l + \underbrace{\eta_n}_{\text{SPONTANEOUS EMISSION}}$$

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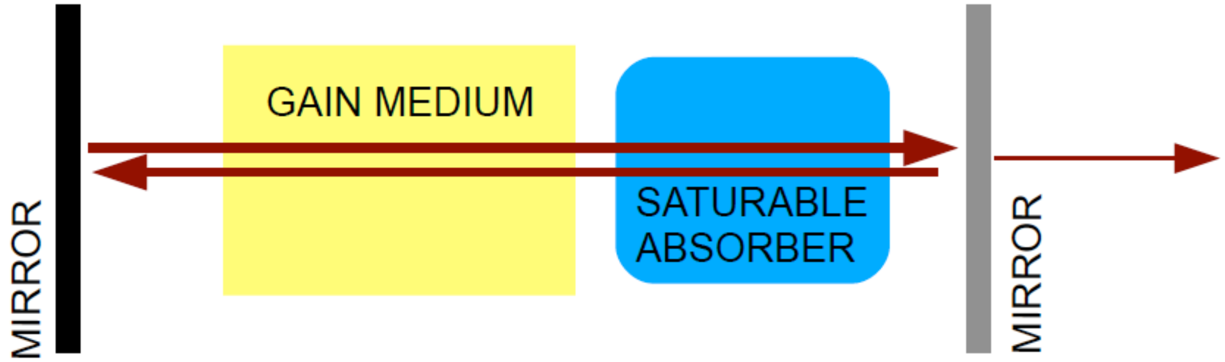

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$$\dot{a}_n = -\imath \frac{\partial \mathcal{H}}{\partial a_n^*} + \eta_n = \imath J_n a_n + \imath J_4 \sum_{\omega_j - \omega_k + \omega_l = \omega_n} a_j a_k^* a_l + \eta_n$$

mode-locking and mode-coupling: no locking -> no coupling

- Theory for random lasers: a mode-locked spin-glass theory (disorder, open cavity)

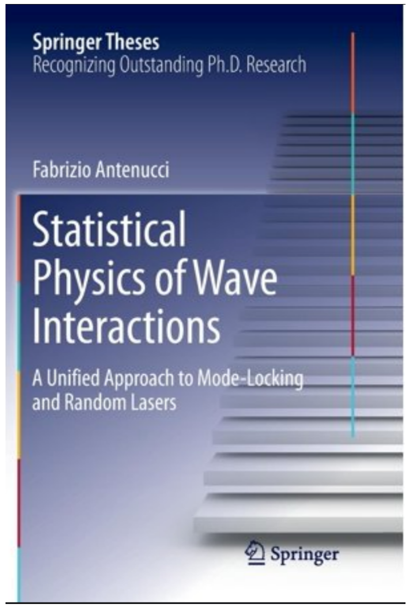
Random laser mode-locked spherical 2+4 phasors' Hamiltonian

Potential, equilibrium-like distribution at inverse temperature $\mathcal{P}^2 = \frac{\epsilon^2}{T} = \frac{1}{T}_{\text{photonic}}$

$$\mathcal{H} = -\sum_{n_1,n_2} J_{n_1 n_2} a_{n_1} a^*_{n_2} - \sum_{\omega_{n_1} - \omega_{n_2} + \omega_{n_3} - \omega_{n_4} = 0} J_{\vec{n}_4} a_{n_1} a^*_{n_2} a_{n_3} a^*_{n_4} + \text{ c.c.}$$

Gain saturation $\longleftrightarrow$ $\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$

Effective interaction



Mode-locking

$$\omega_{n_1} - \omega_{n_2} + \omega_{n_3} - \omega_{n_4} = 0.$$

The mode frequencies must satisfy this condition in order for their mutual interaction to be non-zero.

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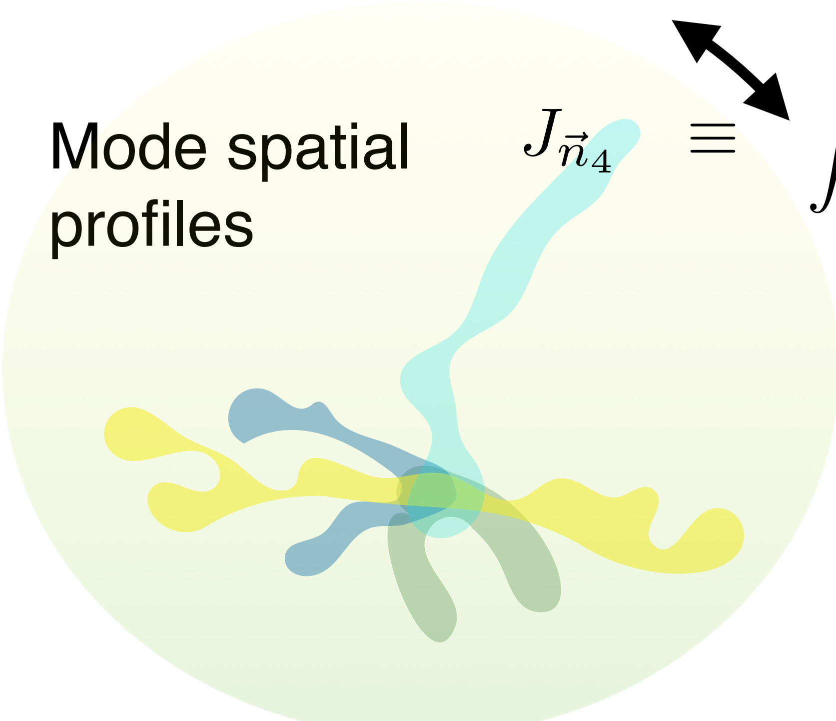


$$J_{\vec{n}_4} \equiv \int d\boldsymbol{r}\ E^{\alpha_1}_{n_1}(\boldsymbol{r})\ E^{\alpha_2}_{n_2}(\boldsymbol{r})\ E^{\alpha_3}_{n_3}(\boldsymbol{r})\ E^{\alpha_4}_{n_4}(\boldsymbol{r})\ \chi^{(3)}_{\vec{\alpha}_4}(\omega_{n_1}, \omega_{n_2}, \omega_{n_3}, \omega_{n_4}; \boldsymbol{r})$$

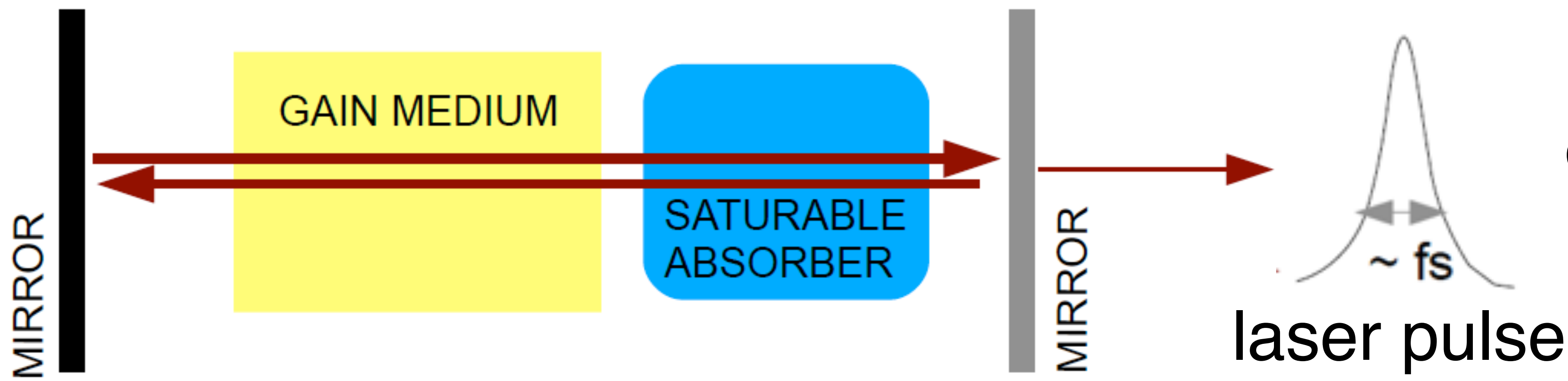
The "interaction" among phasors' expresses the overlap of the spatial profiles of the e.m. modes, modulated by the nonlinear (random) optical susceptibility.
If the modes do not overlap they do not compete for the energy pumped into the system.

Mode-locking

$$\omega_{n_1} - \omega_{n_2} + \omega_{n_3} - \omega_{n_4} = 0.$$

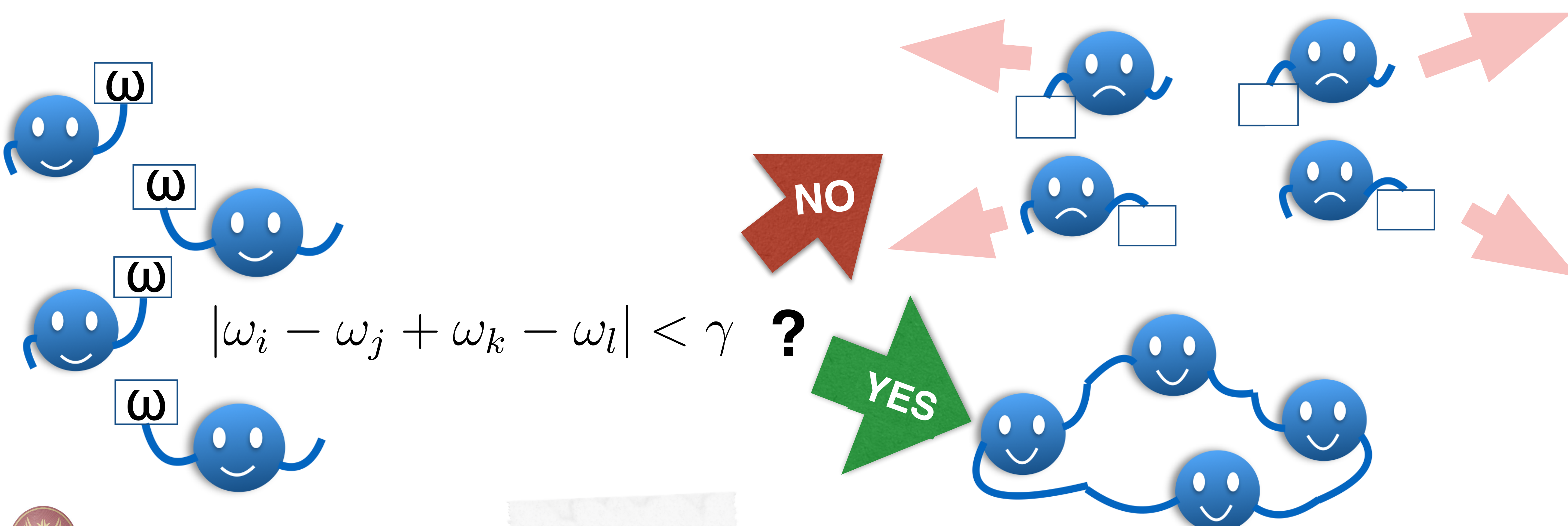
The mode frequencies must satisfy this condition in order for their mutual interaction to be non-zero.

- Theory for ultrafast mode-locked multimode lasers (order, closed cavity)



many modes of different frequencies compose the pulse

these **modes** are ***locked,*** for any set of modes (i,j,k,l) the **frequency matching condition** holds: $|\omega_i - \omega_j + \omega_k - \omega_l| < \gamma$

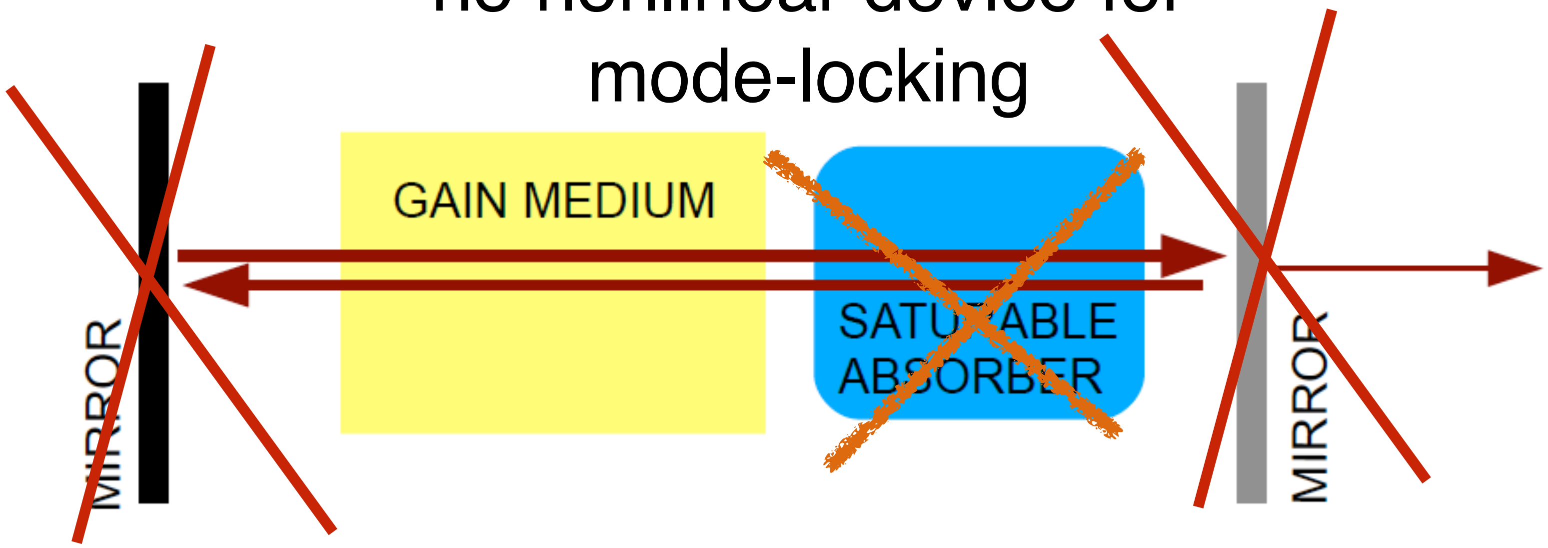


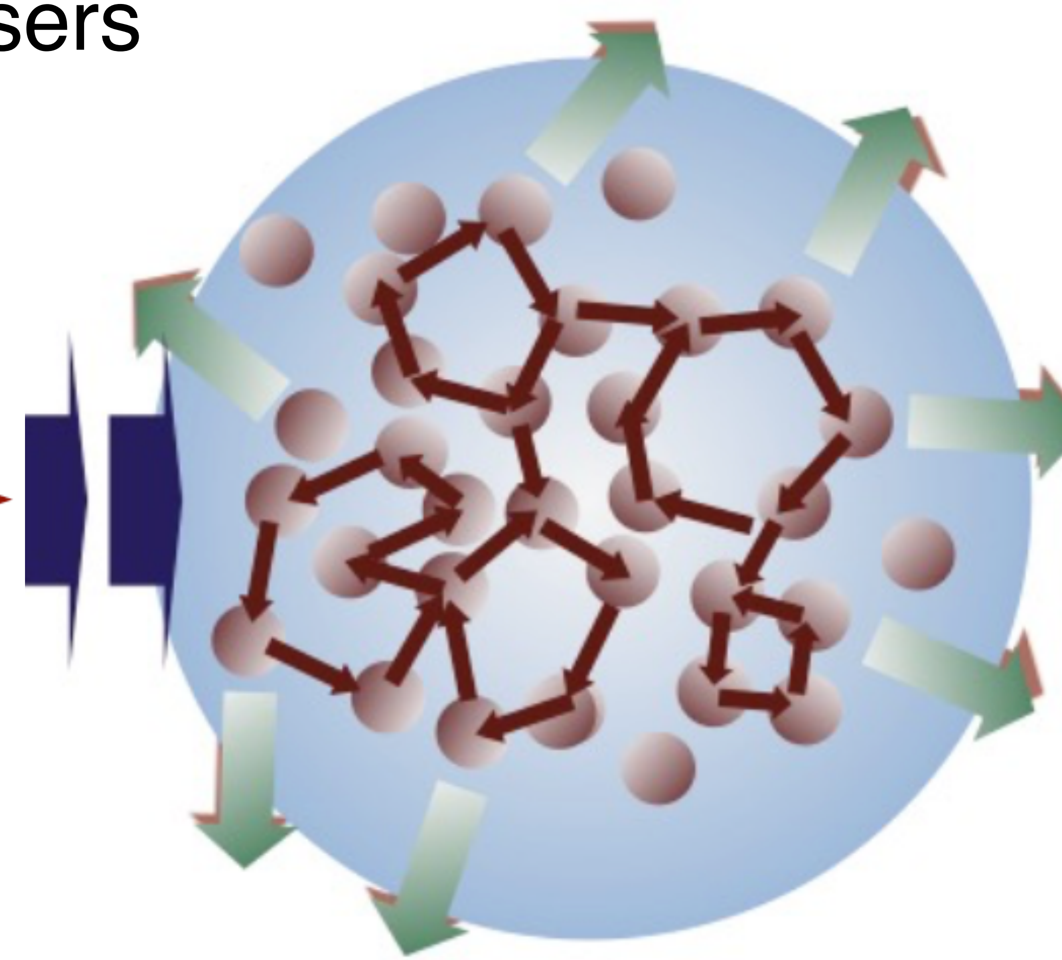
mode-locking and mode-coupling: no locking -> no coupling

MODE LOCKING

Mode-locking in random lasers

no nonlinear device for mode-locking





no mirrors

multiple scattering in place of mirror reflection

mode space overlap and heterogeneous non-linear optical response induces the mode-coupling

no ad hoc device:
self-starting mode-locking in random lasers

F. Antenucci, G. Lerario, B. Silva Fernandez, L. De Marco, M. De Giorgi, D. Ballarini, D. Sanvitto, and LL, Phys. Rev. Lett. 126, 173901 (2021).

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Outline

- Standard and random lasers
- Statistical physics approach to laser physics
  Theory for ultrafast mode-locked multimode lasers (order, closed cavity)
  Theory for random lasers: a mode-locked spin-glass theory (disorder, open cavity)
- The narrow-band solution, phase diagrams, replica symmetry breaking, a new overlap: intensity fluctuation overlap
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps
- In between theory and experiment: a mode-locking model
  Monte Carlo dynamics simulation with exchange Monte Carlo, GPU parallel computing
- Power distribution among modes in the glassy light regime: condensation vs equipartition at high pumping
- Outlook (work in progress)

Random laser mode-locked spherical 2+4 phasors' Hamiltonian *Narrow-band*, aka fully connected

$$\mathcal{P}^2 = \frac{\epsilon^2}{T} = \frac{1}{T}\ \mathrm{photonic}$$

$$\mathcal{H} = -\sum_{n_1,n_2} J_{n_1 n_2} a_{n_1} a^*_{n_2} - \sum_{\omega_{n_1}-\omega_{n_2}+\omega_{n_3}-\omega_{n_4}=0} J_{\vec{n}_4} a_{n_1} a^*_{n_2} a_{n_3} a^*_{n_4} + \text{c.c.}$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

It is exactly solvable in the narrowband approximation

$$\omega_n \simeq \omega_0 \qquad \forall n$$

the mode locking relation

$$\omega_{n_1} - \omega_{n_2} + \omega_{n_3} - \omega_{n_4} = 0.$$

is always satisfied and the interaction network is fully connected

At high pumping (low T)
Replica Symmetry Breaking phases
(one step, full, 1+full)

F Antenucci, C Conti, A Crisanti, LL, PRL 114, 043901 (2015)
F. Antenucci, A. Crisanti, and LL, PRA 91, 053816 (2015)

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Narrow-band random laser spherical 2+4 phasors' Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{jk}^{1,N} J_{jk} a_j a_k^* - \frac{1}{4!}\sum_{j<k<l<m}^{1,N} J_{jklm}\, a_j a_k^* a_l a_m^* + \text{c.c.}\,,$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

The narrow-band solution:

- order parameters,
- phase diagram,
- a new overlap: intensity fluctuation overlap,
- replica symmetry breaking

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ORDER PARAMETERS

$$m_\alpha = \frac{1}{N}\sum_k a_k^\alpha$$

$$q_{\alpha\beta} = \frac{1}{N}\sum_k a_k^\alpha a_k^{*\beta}$$

$$s_{\alpha\beta} = \frac{1}{N}\sum_k a_k^\alpha a_k^\beta$$

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INTENSITY COHERENCE m

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It can be chosen real without loss of generality

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INTENSITY COHERENCE m

COMPLEX AMPLITUDES OVERLAP $q_{\alpha\beta}$

REPLICA SYMMETRY BREAKING

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$$s_{\alpha\beta} = \frac{1}{N}\sum_k a_k^\alpha a_k^\beta$$

The imaginary part can be set to zero, the off-diagonal real part is equal to $q_{\alpha\beta}$, the diagonal real part is r_d

INTENSITY COHERENCE m

PHASE COHERENCE r_d

COMPLEX AMPLITUDES OVERLAP

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PHASE DIAGRAMS

$$\mathcal{P} = \epsilon\sqrt{\beta J_0}$$

PUMPING RATE (INVERSE TEMPERATURE)

$$R_J \equiv \frac{J}{J_0}$$

DISORDER DEGREE (msd / mean)

$$J_4^2 = \alpha^2 J^2, \quad J_2^2 = (1-\alpha)^2 J^2,$$

“CLOSEDNESS”

F. Antenucci, A. Crisanti, and LL, PRA 91, 053816 (2015)

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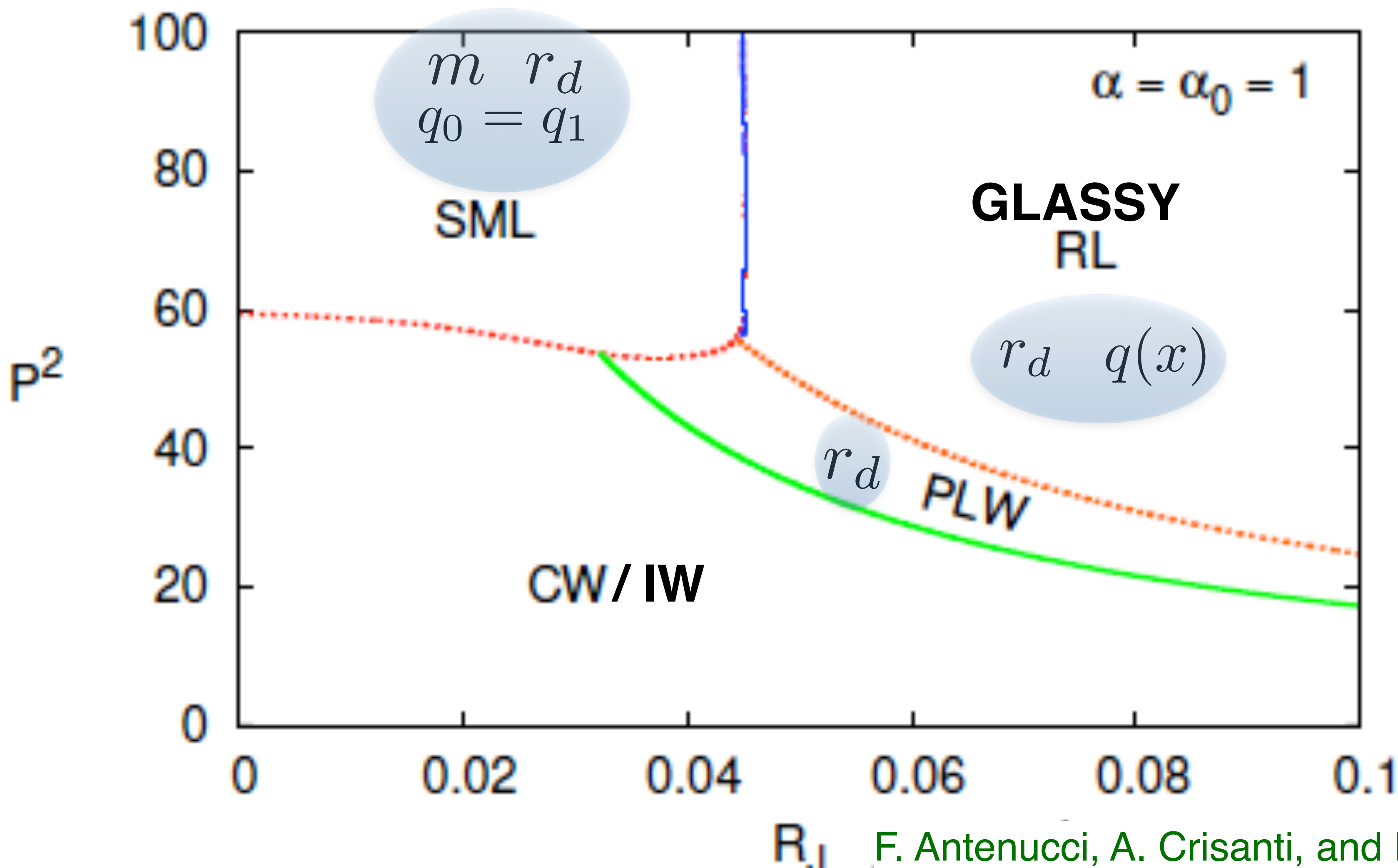
PHASE DIAGRAMS

CLOSED CAVITY + ANY DEGREE OF DISORDER

SML: Standard (and random) ML laser
CW/IW: Continuous/Incoherent Wave regime
PLW: Phase-Locked Wave regime
GRL: Gassy Random Laser

SML ↔ (random) Ferromagnet
CW/IW ↔ Paramagnet
PLW ↔ ----
GRL ↔ **Glassy phase**

$$R_J \equiv \frac{J}{J_0}$$



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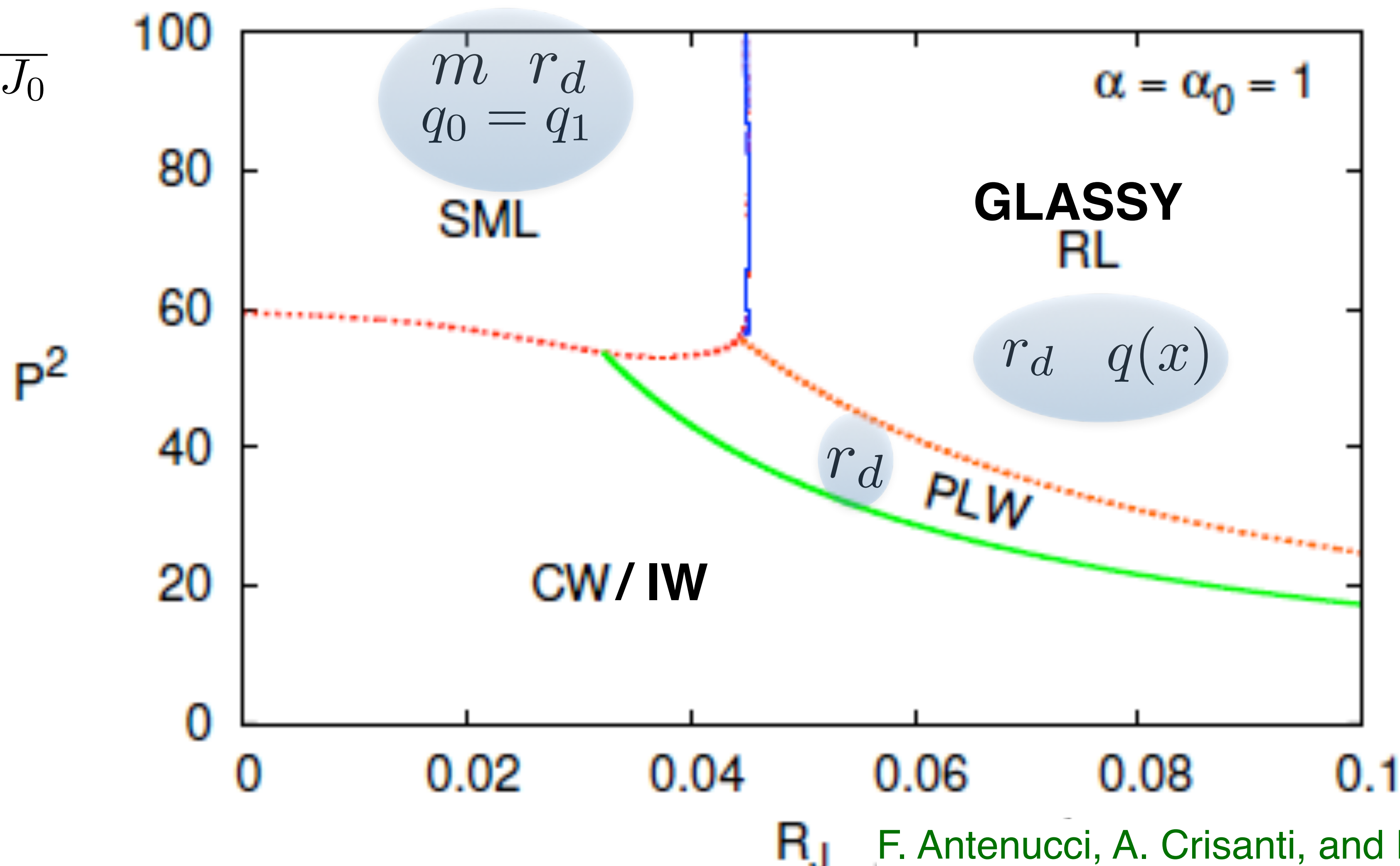

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Narrow-band random laser spherical 2+4 phasors' Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{jk}^{1,N} J_{jk} a_j a_k^* - \frac{1}{4!}\sum_{j<k<l<m}^{1,N} J_{jklm}\, a_j a_k^* a_l a_m^* + \text{c.c.}\,,$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

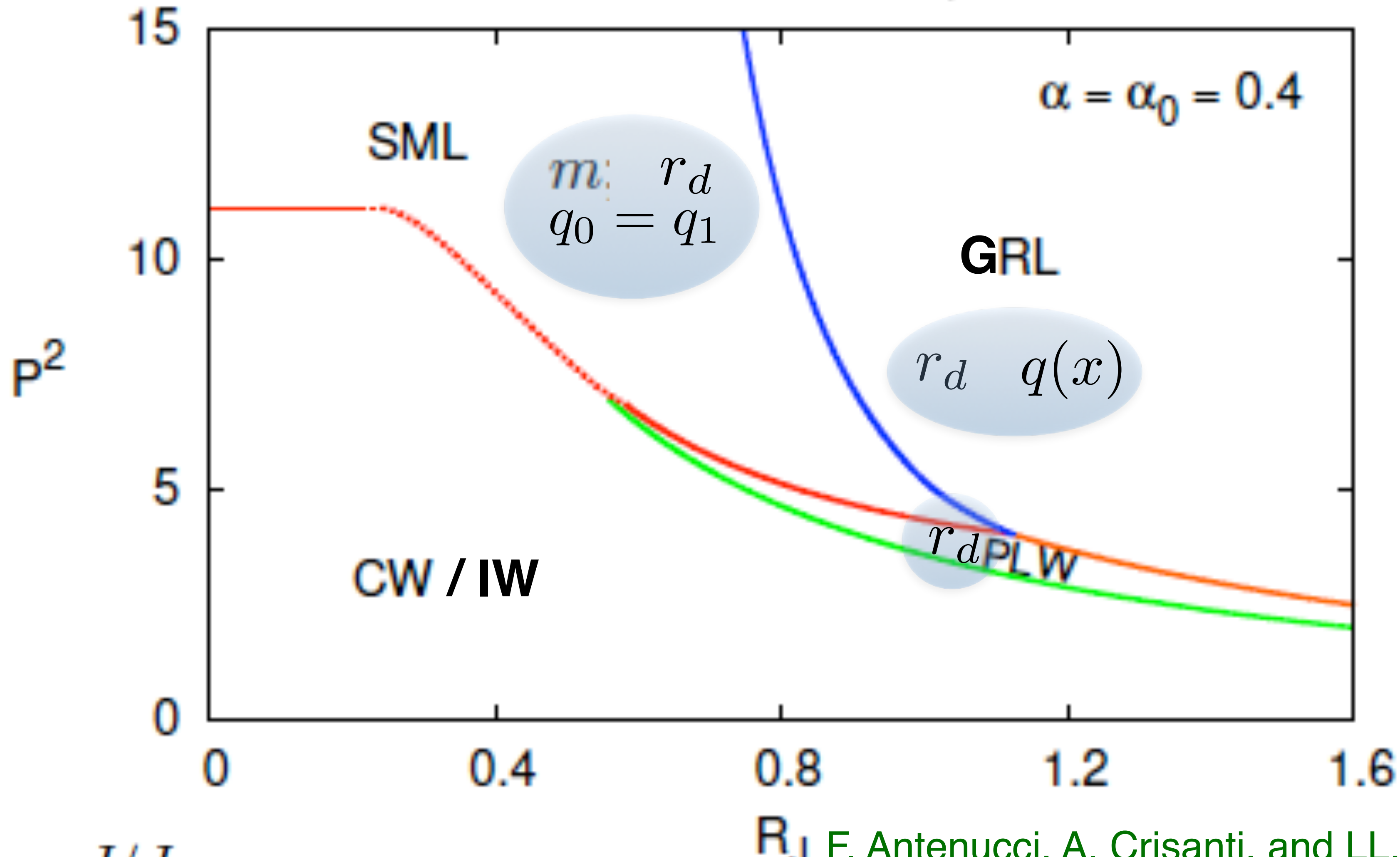
PHASE DIAGRAMS

OPEN CAVITY + ANY DEGREE OF DISORDER

SML: Standard (and random) ML laser
CW/IW: Continuous/Incoherent Wave regime
PLW: Phase-Locked Wave regime
GRL: Gassy Random Laser

SML ↔ (random) Ferromagnet
CW/IW ↔ Paramagnet
PLW ↔ ----
GRL ↔ **Glassy phase**

$$\mathcal{P} = \epsilon\sqrt{\beta J_0}$$



$$J_4^2 = \alpha^2 J^2, \quad J_2^2 = (1-\alpha)^2 J^2,$$

$$R_J = J/J_0$$

F. Antenucci, A. Crisanti, and LL, PRA 91, 053816 (2015)

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Narrow-band random laser spherical 2+4 phasors' Hamiltonian

An extra (useful) parameter: IFO
Intensity Fluctuation Overlap

Since our 'spins' are not *locally* bounded we can define an *intensity fluctuation overlap* between replicas

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{N}\sum_{k=1}^{N}\left(|a_k^{(\alpha)}|^2|a_k^{(\beta)}|^2 - \langle|a_k|^2\rangle^2\right)$$

This is the *standard* Parisi overlap in this model: $q_{\alpha\beta} = \frac{1}{N}\sum_k a_k^\alpha a_k^{*\beta}$

In the *fully connected* mean-field mixed spherical phasor model one proves

$$\mathcal{Q}_{ab} = q_{ab}^2 - \frac{|m|^4}{4}$$

There is a one-to-one correspondence between elements of the standard (Parisi) overlap and IFO matrices
In principle ***IFO can be experimentally measured***

F Antenucci, A Crisanti, LL, Sci. Rep. 5, 16792 (2015)

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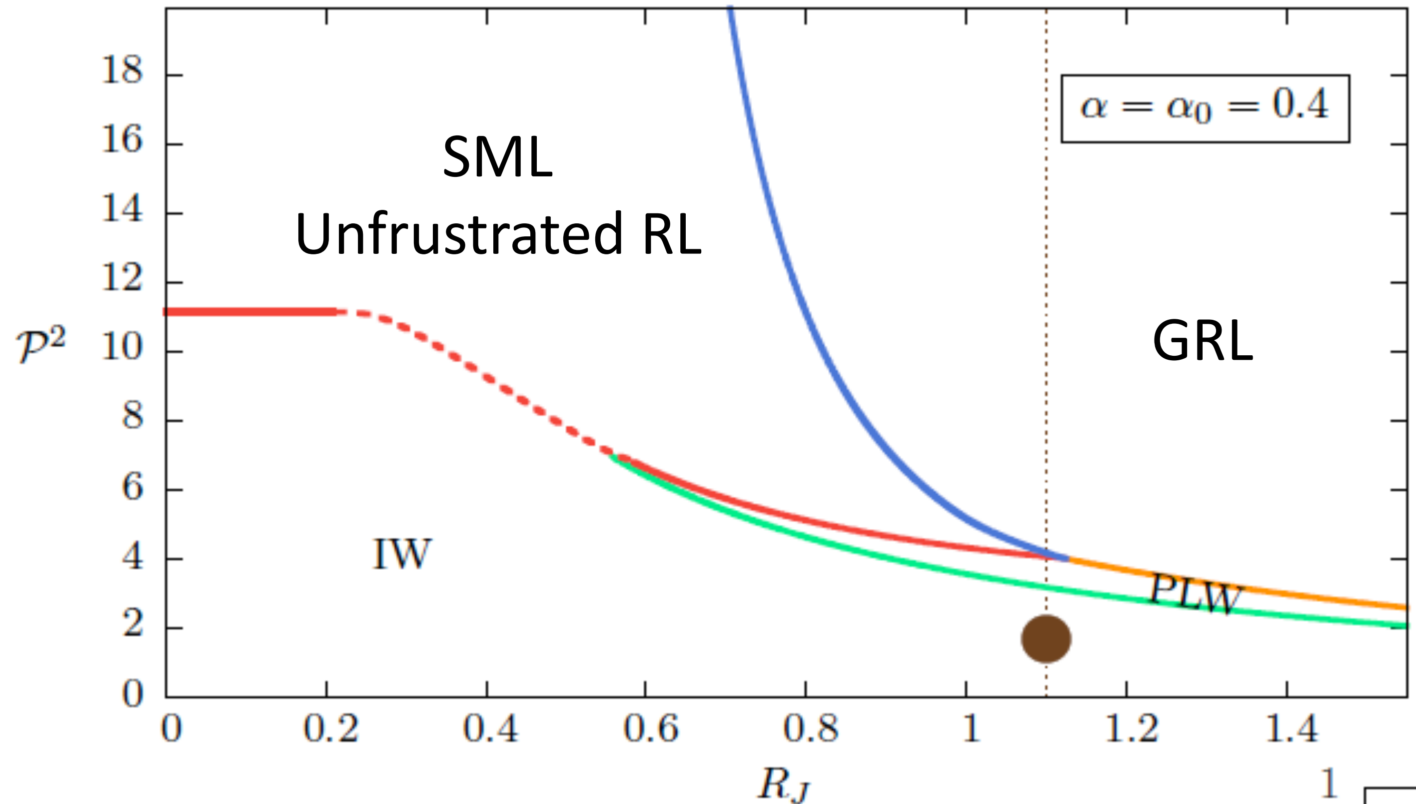
PHASE DIAGRAM

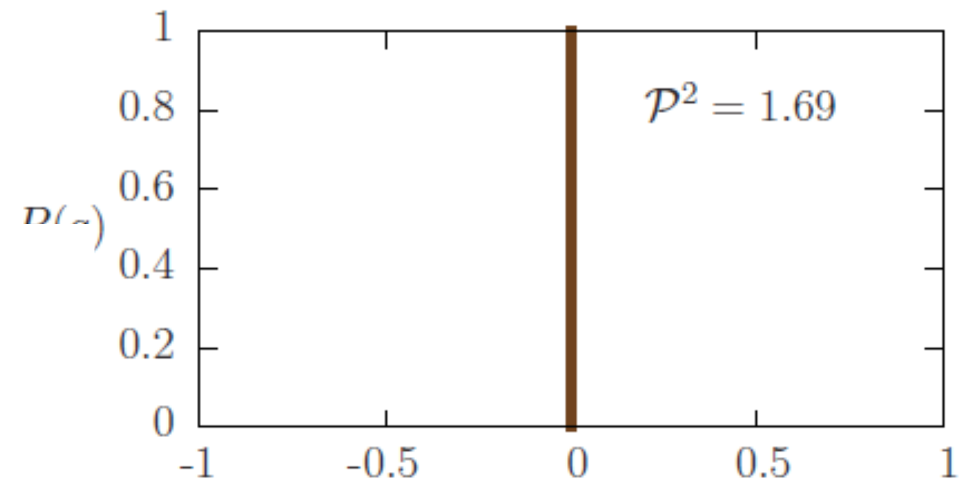
MFT: Intensity Fluctuations Overlap



REPLICA SYMMETRY BREAKING

IFO DISTRIBUTION

Replica Symmetry



$\mathcal{P}(\mathcal{Q})$

F. Antenucci, A. Crisanti, LL, SciRep 2015

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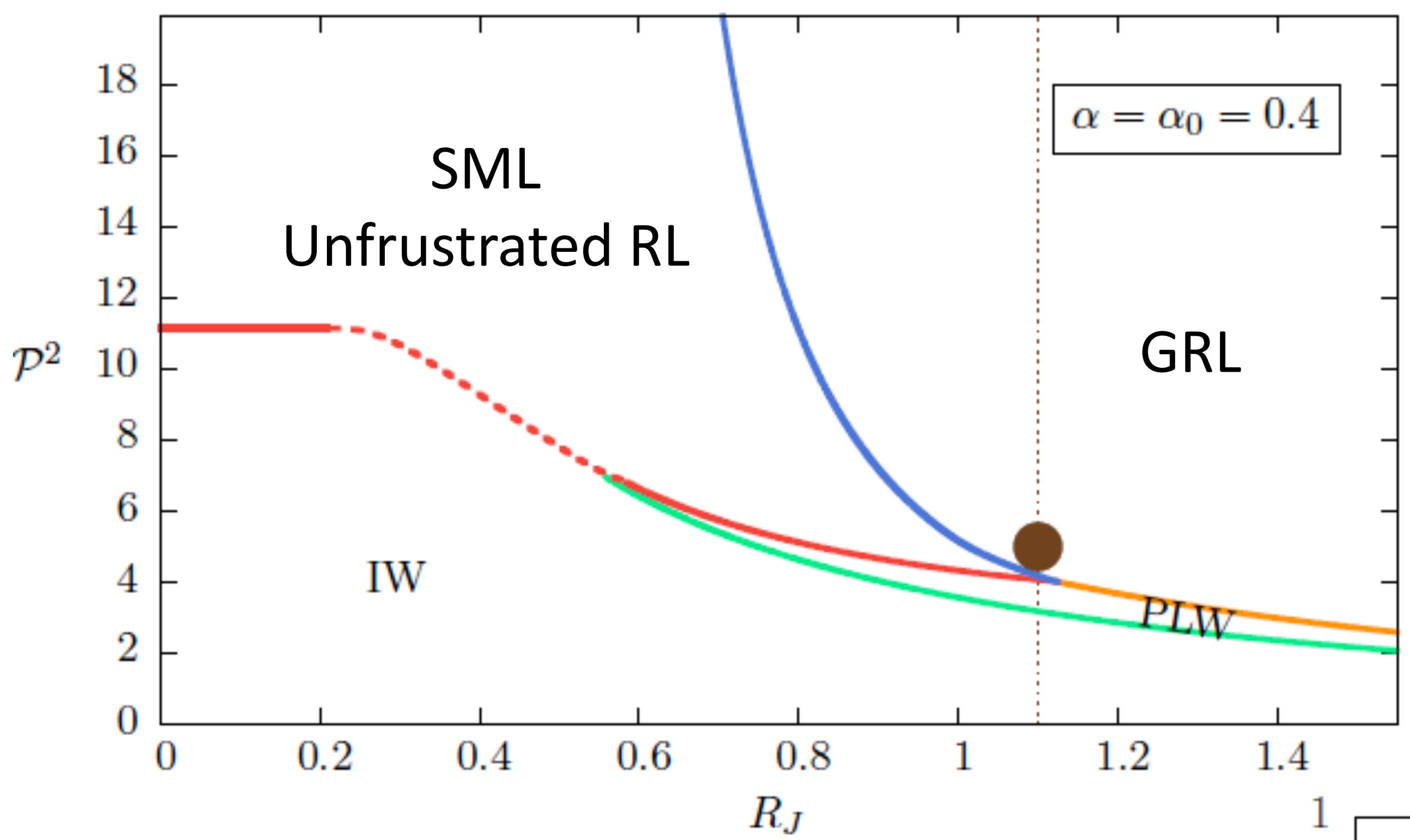
PHASE DIAGRAM

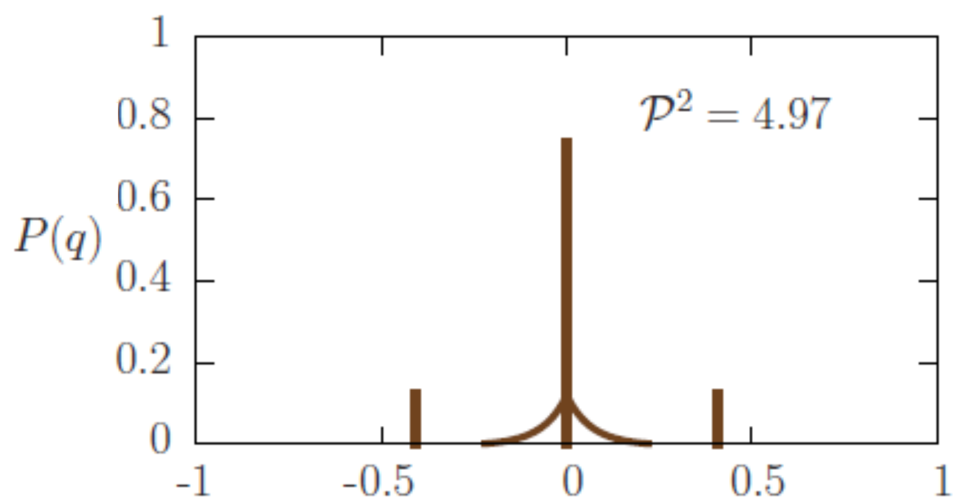
MFT: Intensity Fluctuations Overlap



REPLICA SYMMETRY BREAKING

IFO DISTRIBUTION

1+Full Replica Symmetry Breaking



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F. Antenucci, A. Crisanti, LL, SciRep 2015

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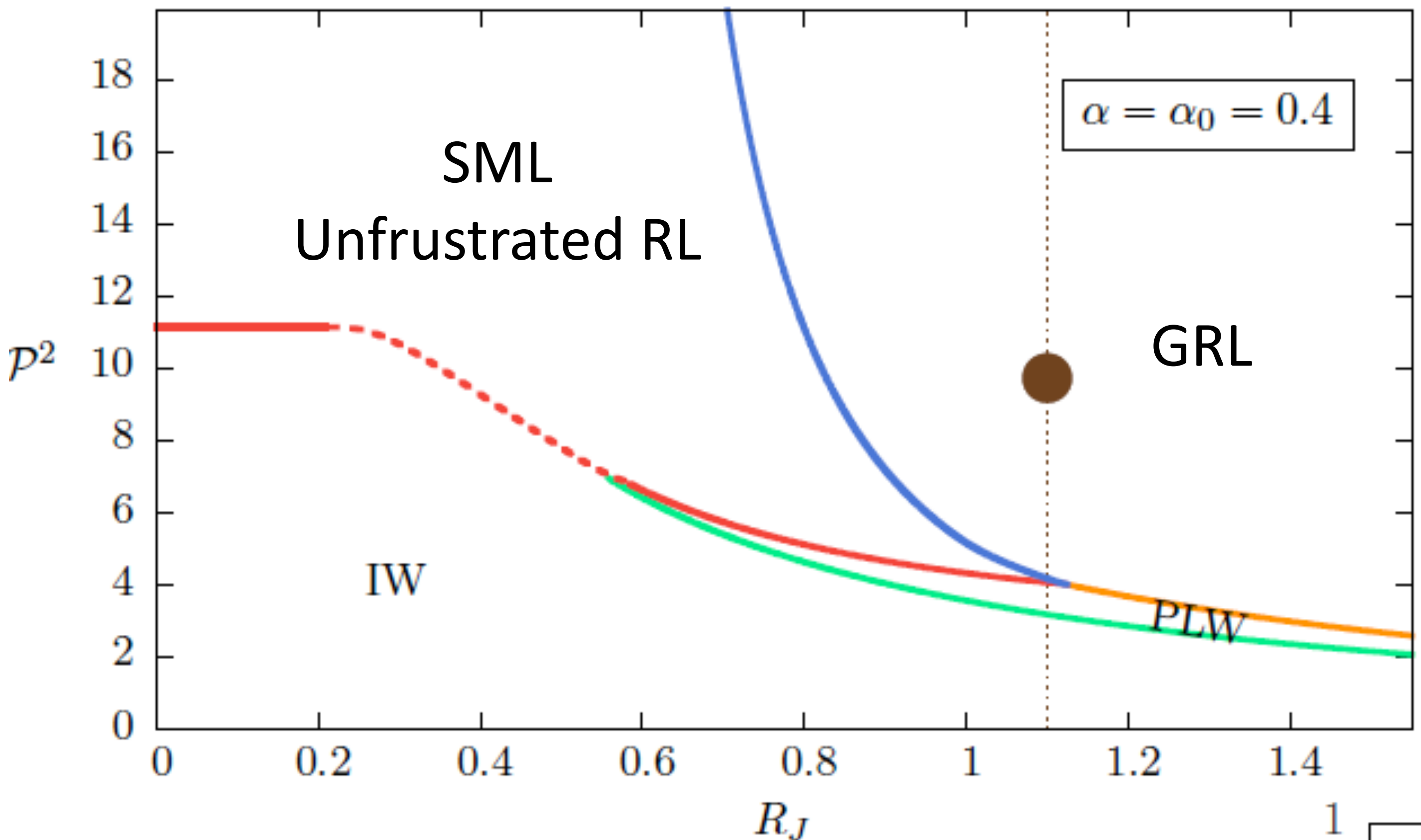
PHASE DIAGRAM

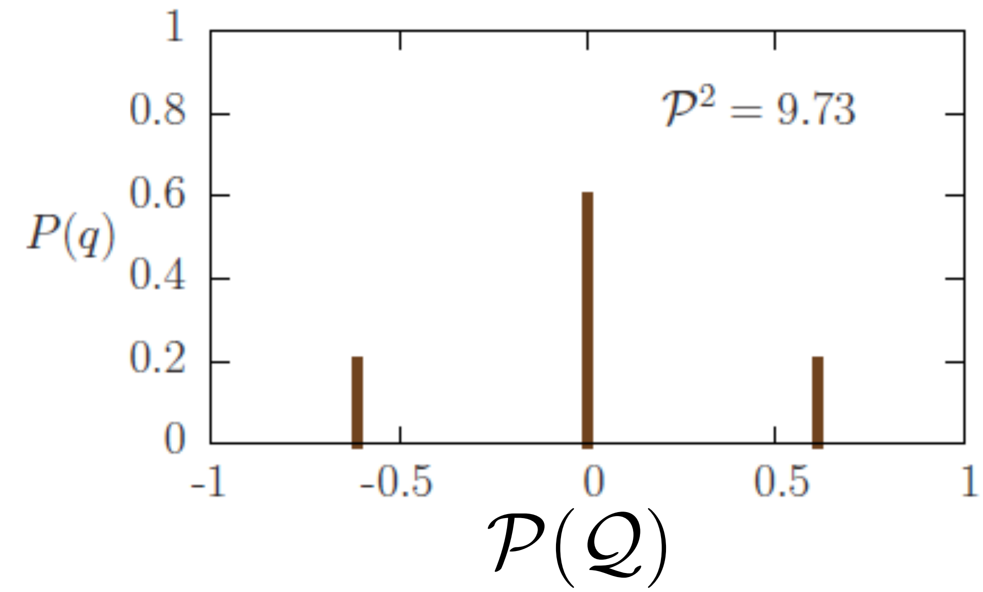
MFT: Intensity Fluctuations Overlap



REPLICA SYMMETRY BREAKING

IFO DISTRIBUTION

1-step Replica Symmetry Breaking



F. Antenucci, A. Crisanti, LL, SciRep 2015

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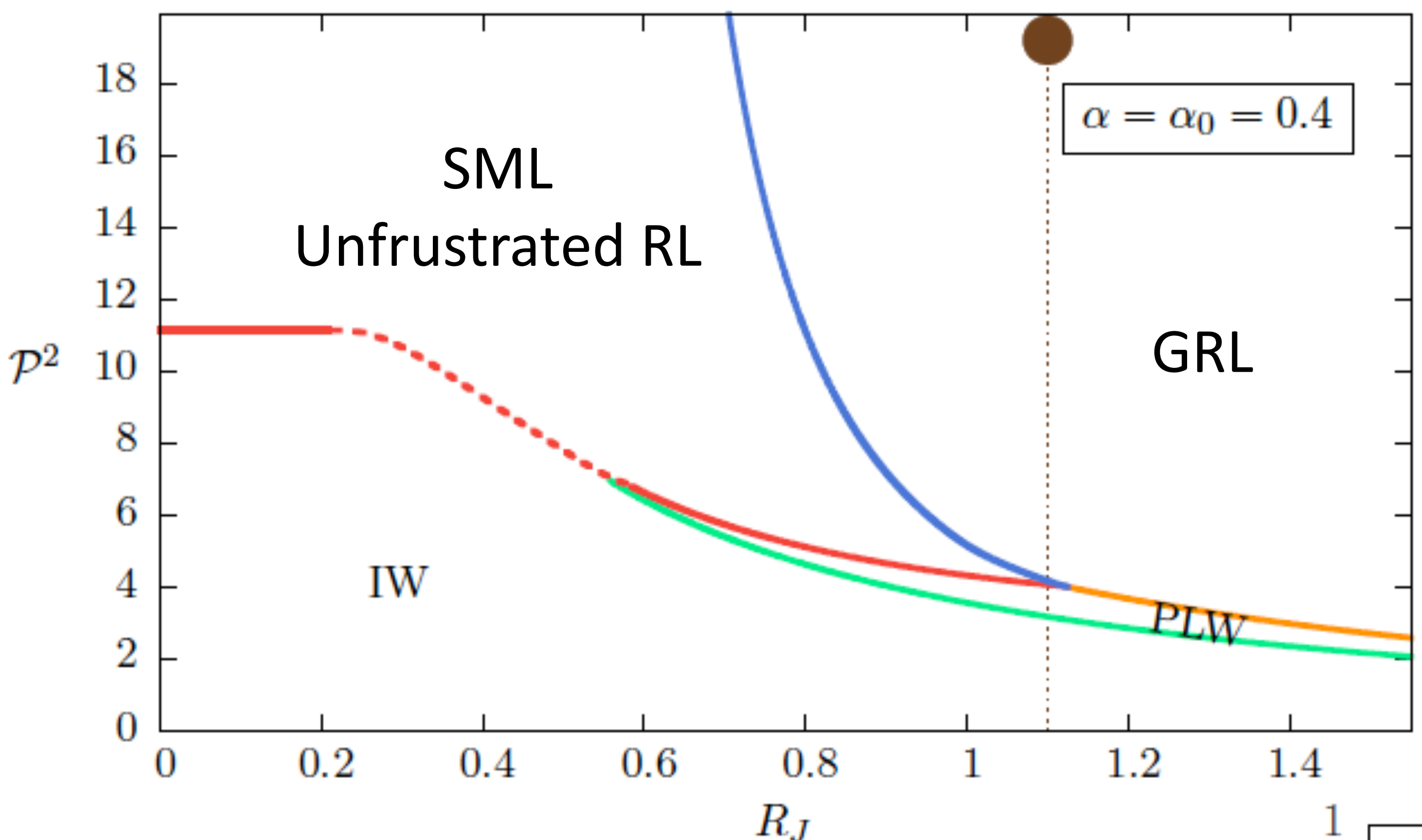
PHASE DIAGRAM

MFT: Intensity Fluctuations Overlap



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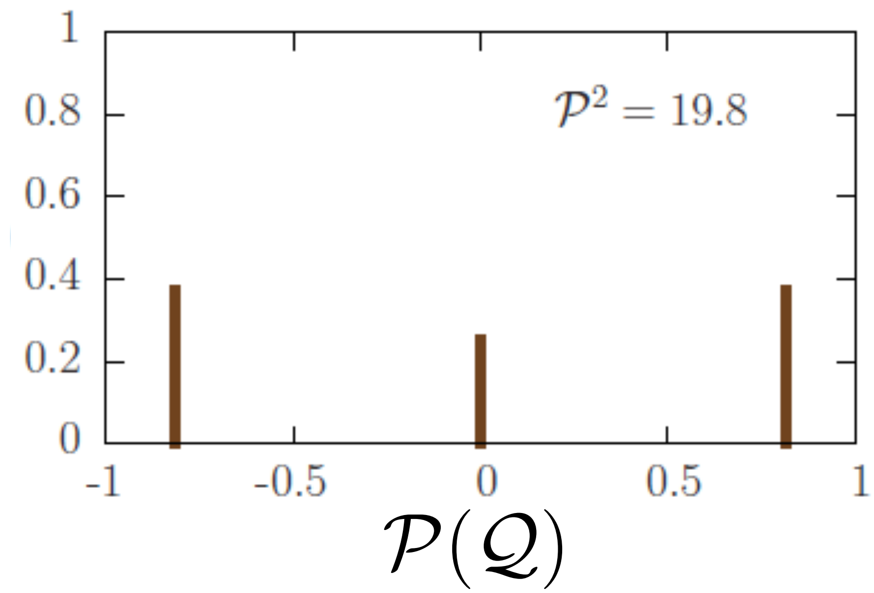


F. Antenucci, A. Crisanti, LL, SciRep 2015

Narrow-band random laser spherical 2+4 phasors' Hamiltonian

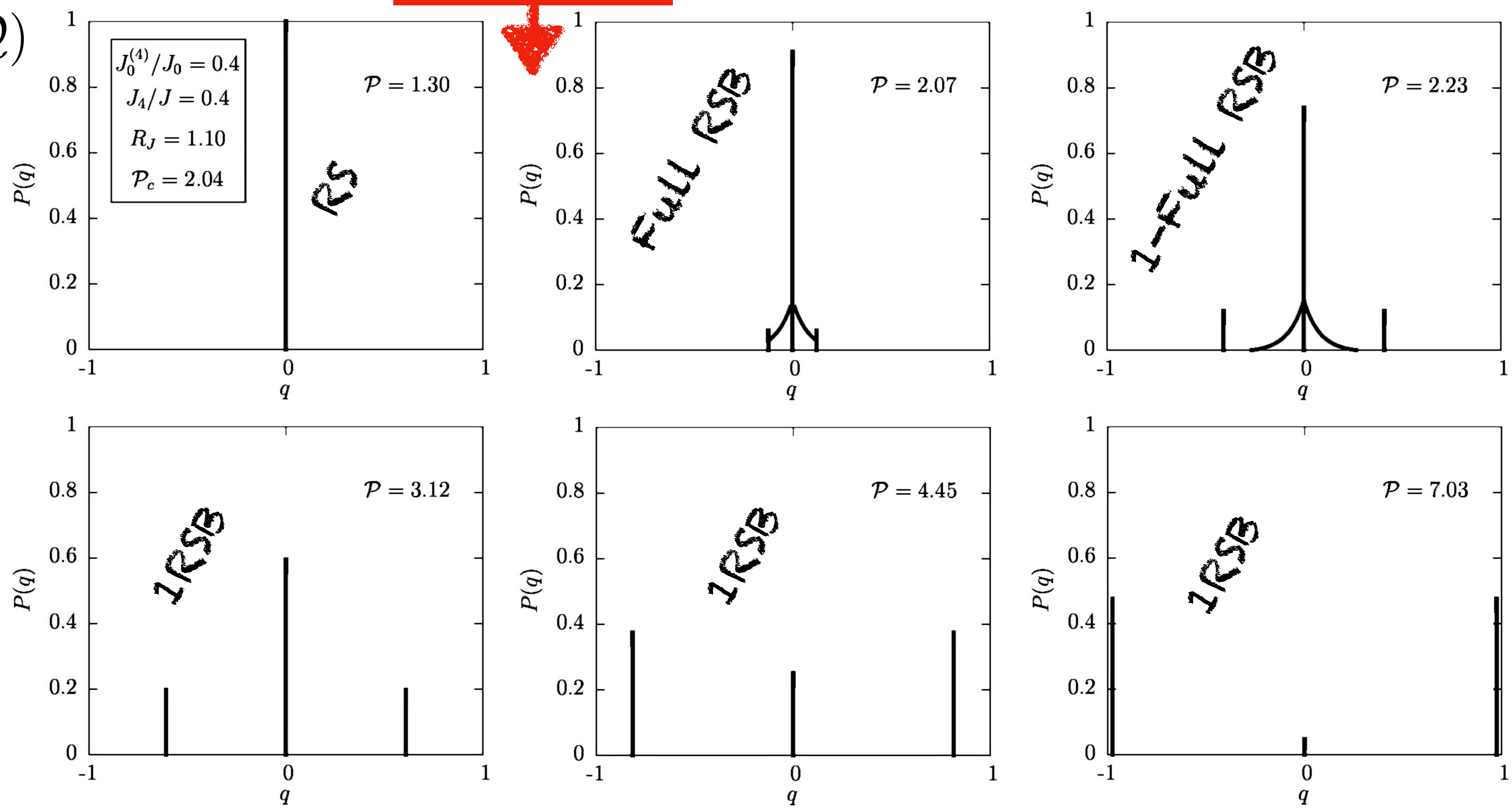
REPLICA SYMMETRY BREAKING

Intensity Fluctuation Overlap probability distribution in the fully connected 2+4 spherical phasor model

F. Antenucci, A. Crisanti, LL, SciRep 2015

RSB transition

$P(\mathcal{Q})$



Narrow-band random laser spherical 2+4 phasors' Hamiltonian

REPLICA SYMMETRY BREAKING

Mean-field $\mathcal{Q}_{ab} = q_{ab}^2$
fully connected theory

F. Antenucci, A. Crisanti, LL, SciRep 2015

Intensity Fluctuation Overlap probability distribution in the fully connected 2+4 spherical phasor model

Element-element relation: any Replica Symmetry Breaking is detectable and measurable.

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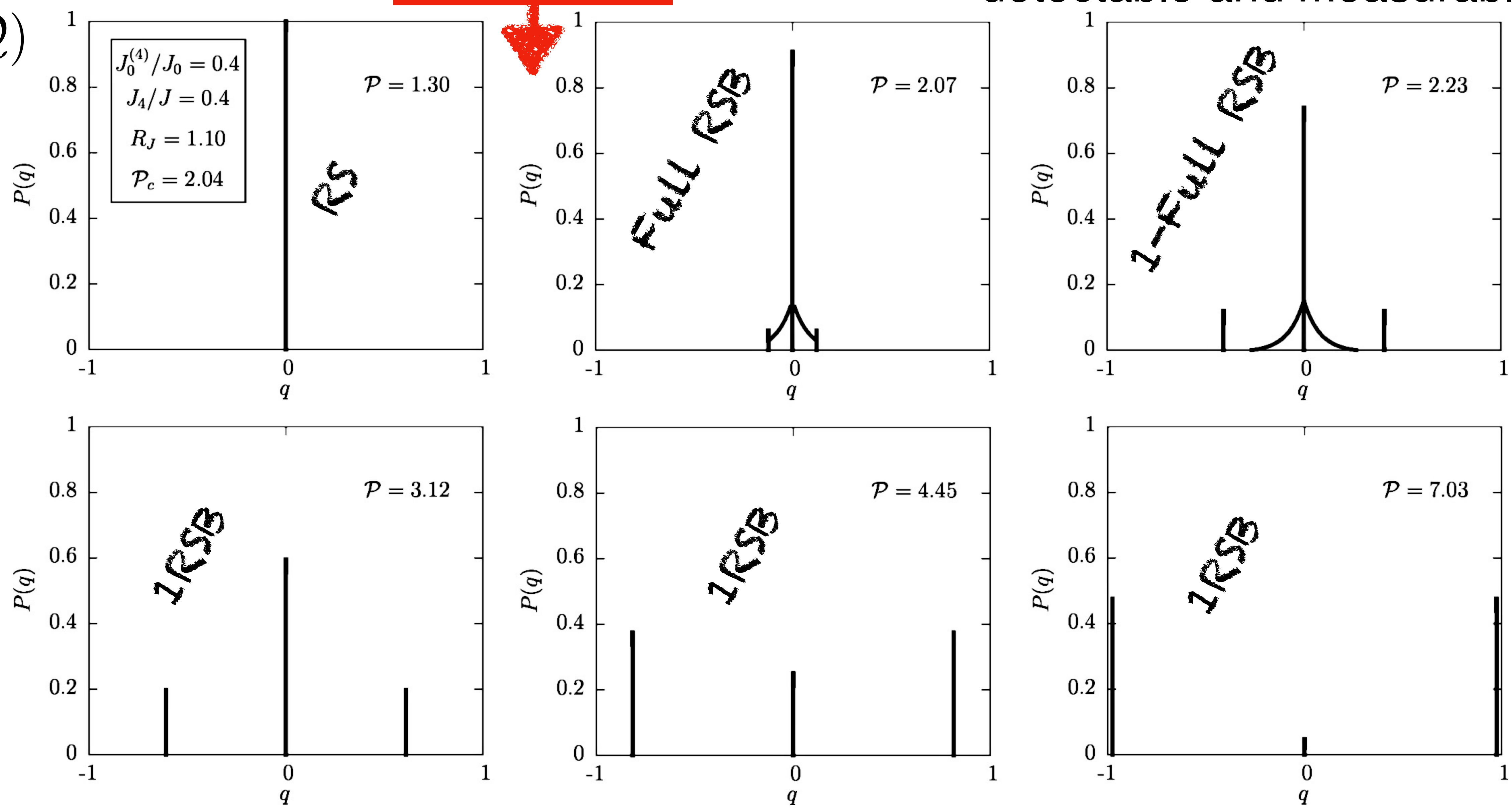

Outline

- Standard and random lasers
- Statistical physics approach to laser physics
  Theory for ultrafast mode-locked multimode lasers (order, closed cavity)
  Theory for random lasers: a mode-locked spin-glass theory (disorder, open cavity)
- The narrow-band solution, phase diagrams, replica symmetry breaking, a new overlap: intensity fluctuation overlap
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps
- In between theory and experiment: a mode-locking model
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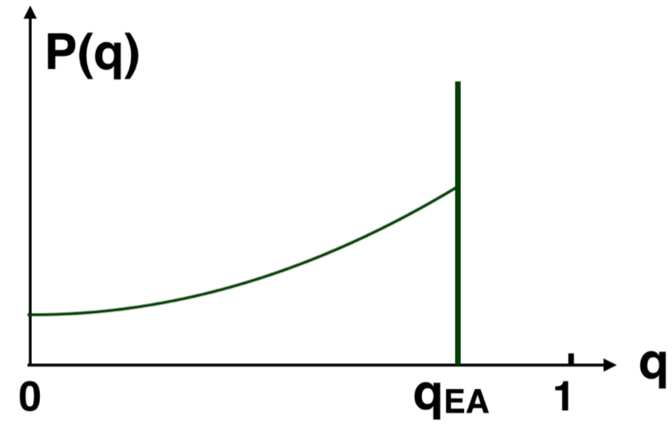
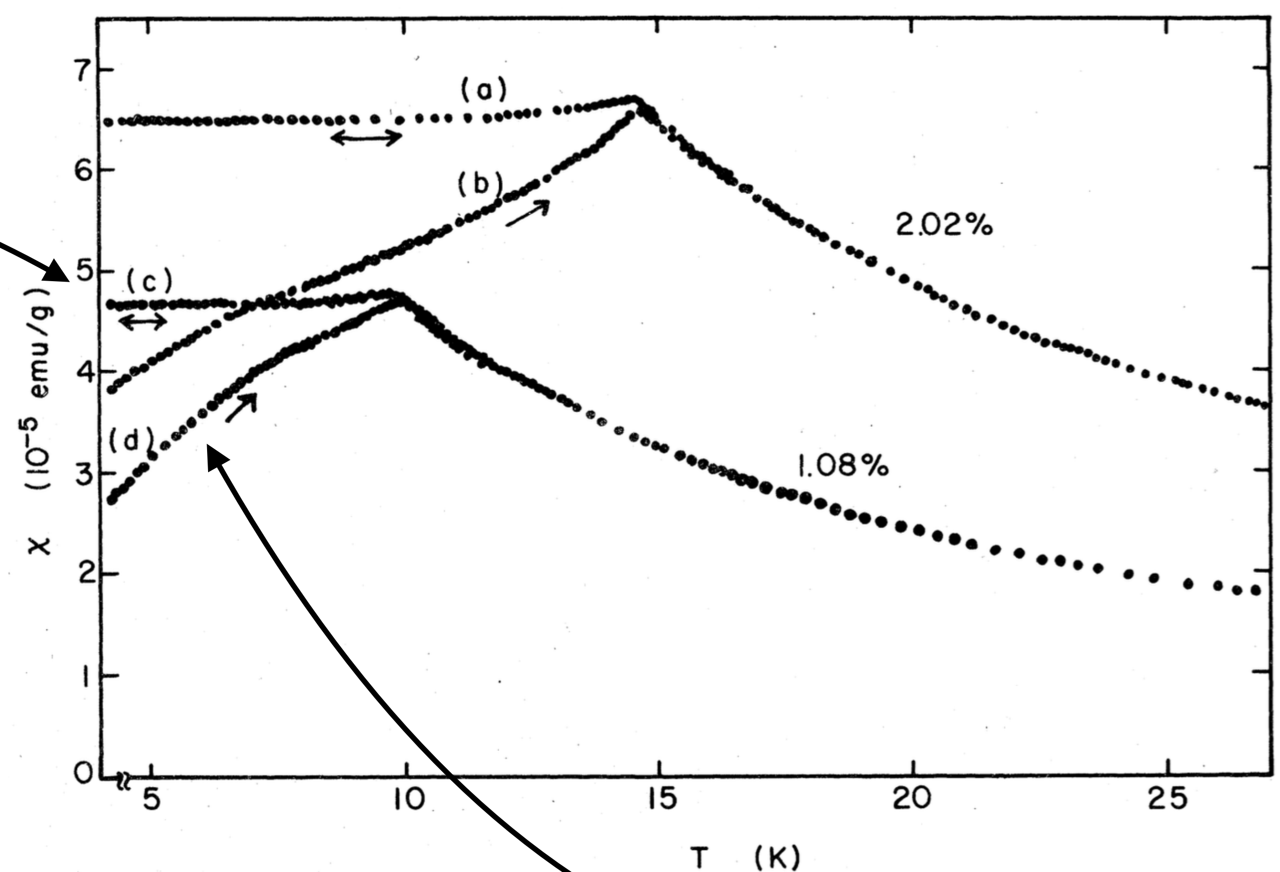
- Outlook (work in progress)

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

The RSB order parameter
is the distribution of the overlap values



What is accessible to experiments?
Static susceptibility after cooling in zero magnetic field or in a field h



Field Cooled
Susceptibility

$$\chi_{\rm FC} = \beta\,(1 - \langle q \rangle)$$

$$\langle q \rangle = \int_0^1 dq\, q\, P(q)$$

Zero Field Cooled
Susceptibility

$$\chi_{\rm ZFC} = \beta\,(1 - q_{\rm EA})$$

$$q_{\rm EA} = \frac{1}{N}\sum_i \overline{m_i^2} = \max_q P(q)$$

Nagata S, Keesom P and Harrison H, Phys. Rev. B 19 1633 (1979)

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

What hinders the whole P(q) measure?

We need microscopic configurations at equilibrium

$$q_{ab} = \frac{1}{N}\sum_{i=1}\sigma_i^{(a)}\sigma_i^{(b)}$$

configuration of replica a at equilibrium

$$\frac{1}{Z}\sum_{\{\sigma^{(a)}\}}\exp\left\{-\beta\mathcal{H}_J[\{\sigma^{(a)}\}]\right\}$$

$$q_{\alpha_1\alpha_2} \equiv \frac{1}{N}\sum_{i=1}^{N} m_i^{(\alpha_1)} m_i^{(\alpha_2)}$$

pure state α
with clustering property

P(q)

Microscopic atomic spin configurations in spin glasses are hard to be measured

Equilibrium is hardly/never attained in experiments on spin-glasses

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

Statics-dynamics equivalence

$$X(q) = \lim_{\substack{t,t'\to\infty \\ C(t,t')=q}} \frac{\partial \chi(t,t')}{\partial t'} \Big/ \frac{\partial C(t,t')}{\partial t'}$$

Fluctuation-Dissipation Ratio

Cumulative distribution of overlap $\tilde{X}(q) = \int_0^q dq' \, \tilde{P}(q')$

Franz S, Mézard M, Parisi G, Peliti L, Phys. Rev. Lett. 81 1758 (1998)

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

Statics-dynamics equivalence

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Fluctuation-Dissipation Ratio

Cumulative distribution of overlap $\tilde{X}(q) = \int_0^q dq'\, \tilde{P}(q')$

If the dynamic FDR can be measured in experiments
and if the system is **stochastically stable***

$$X(q) = \tilde{X}(q)$$

and we can recover the Parisi distribution

*A reshuffling of pure states takes place in the (limit) procedure of the costruction of $\tilde{P}$(q) in presence of ergodicity breaking. If such <u>reshuffling only lift the degeneracy due to symmetries</u> of the system, then $\tilde{P}$(q) is the right distribution of the overlap.

Franz S, Mézard M, Parisi G, Peliti L, Phys. Rev. Lett. 81 1758 (1998)

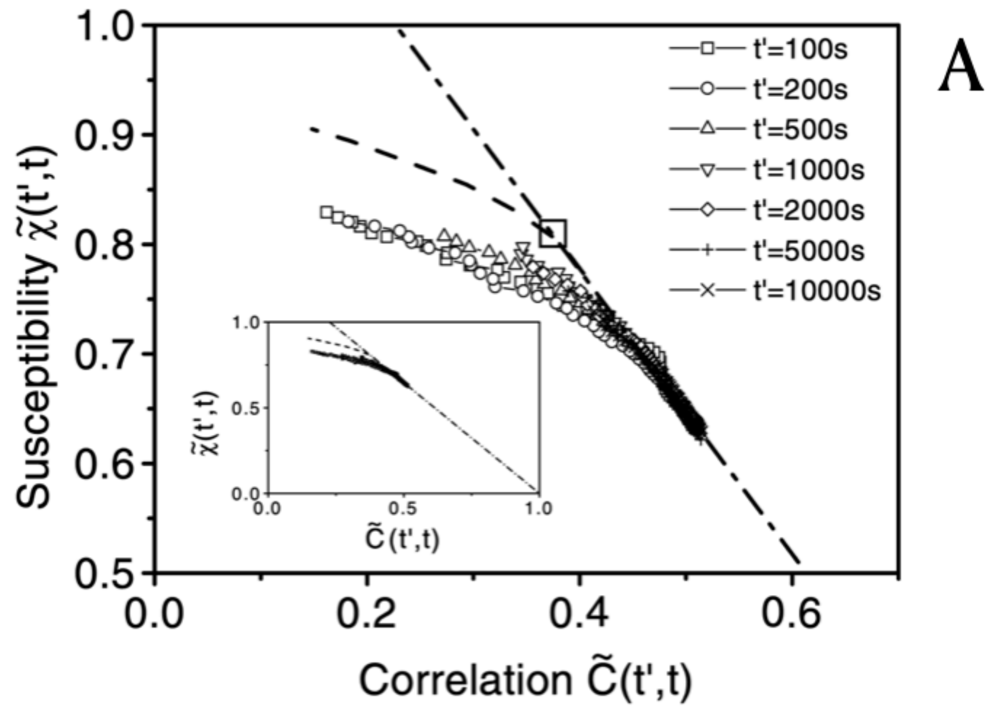
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

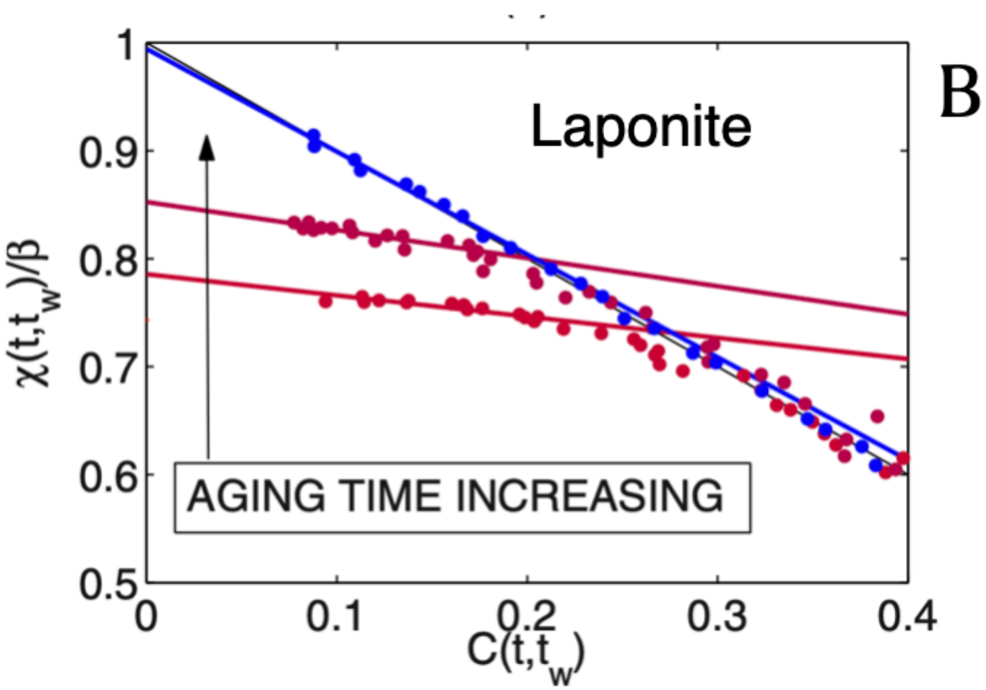
Statics-dynamics equivalence

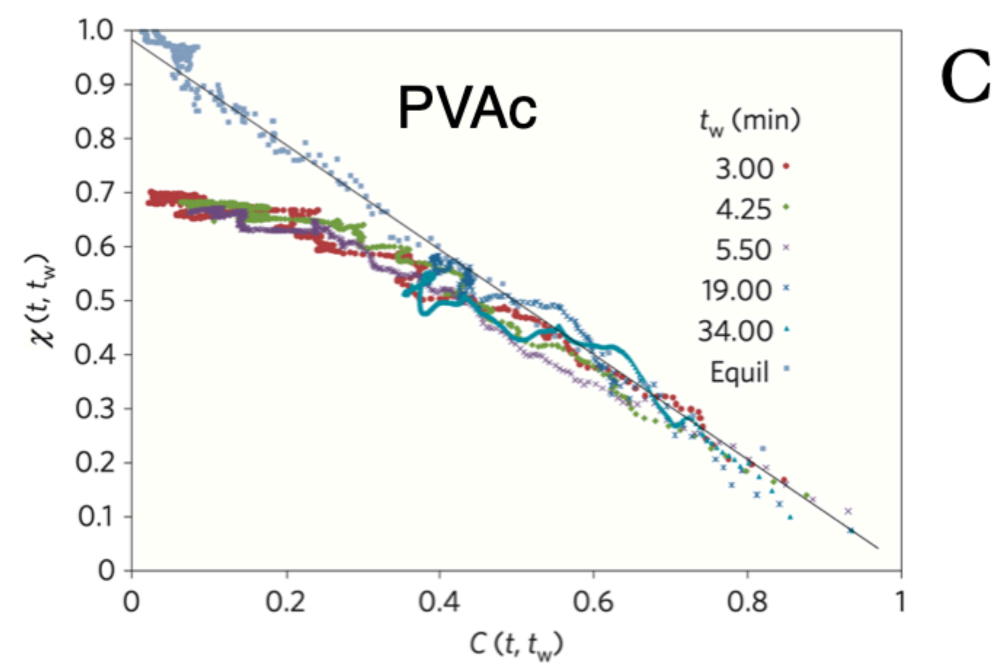
FDR measure is harder than expected in real glassy systems

$Cd\,Cr_{1.7}\,In_{0.3}\,S_4$
spin-glass

Herisson, Ocio, PRL 88, 257202 (2002)







Maggi et al, PRB 81, 104201 (2010)

Ourkis, Israeloff, Nature Phys 6, 135 (2010)

C Conti, N Ghofraniha, LL, G Ruocco, in "Spin Glass Theory and Far Beyond: Replica Symmetry Breaking After 40 Years", pp. 307-334 (World Scientific, 2023), arXiv:2209.03781

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Herisson, Ocio, PRL 88, 257

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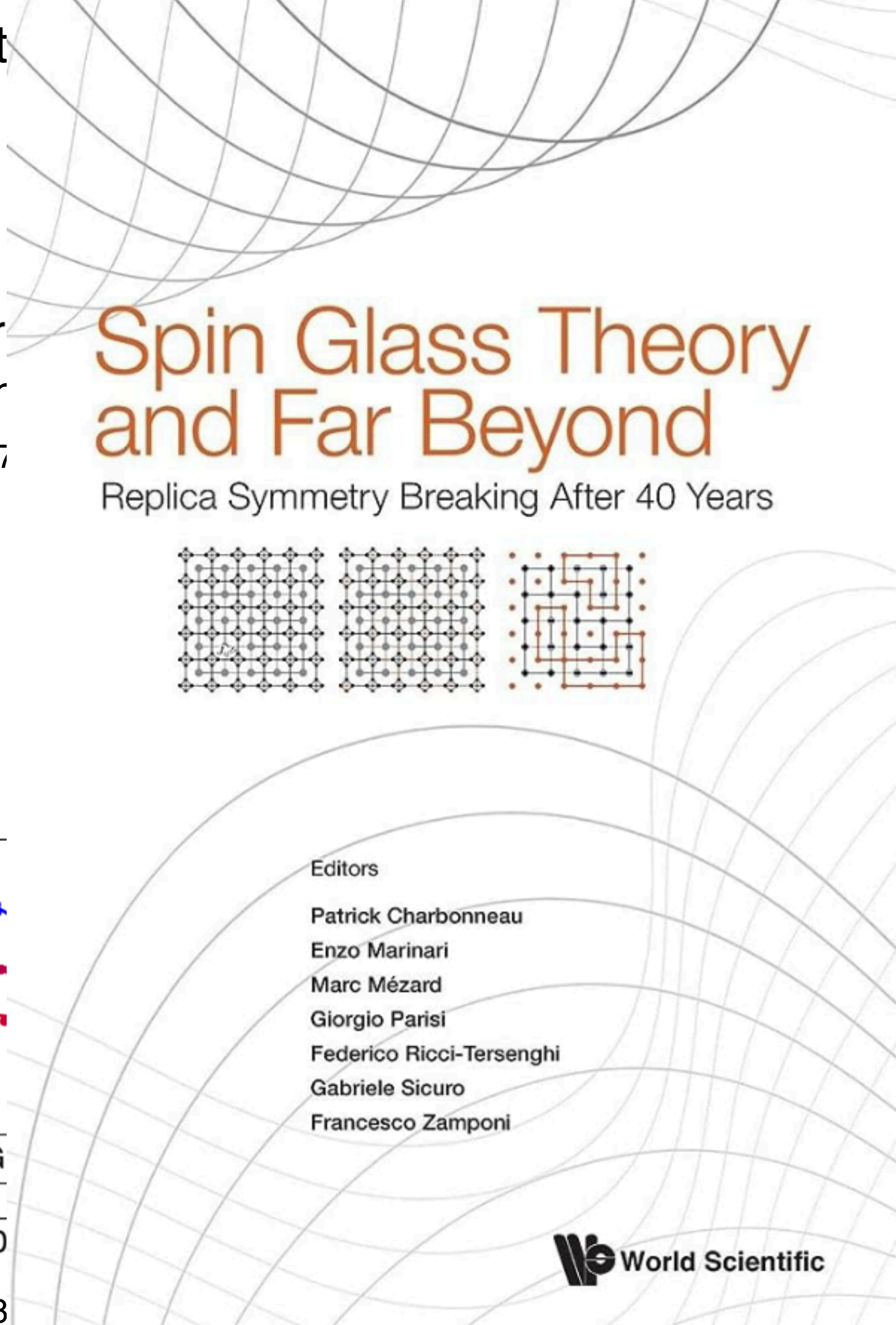
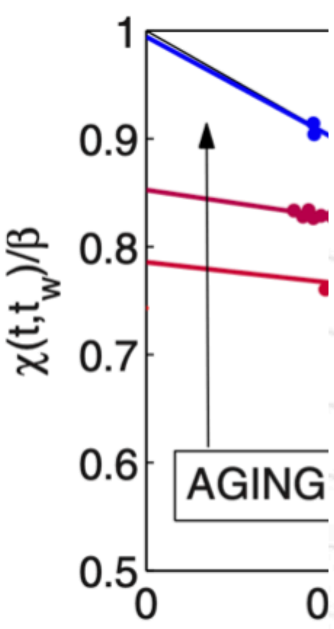
arXiv:2209.03781

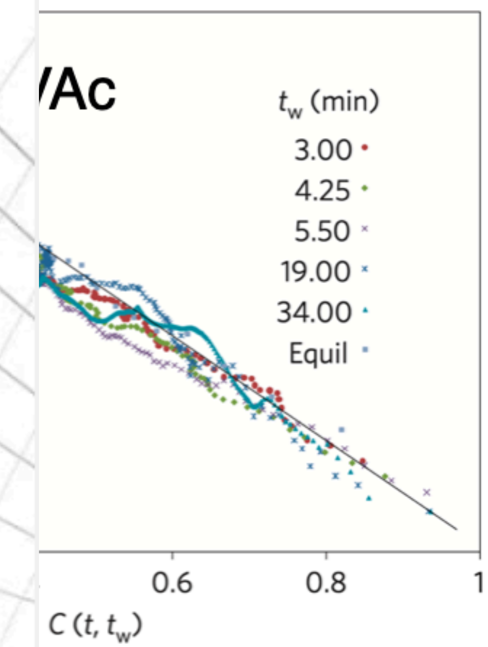


FDR measure is harder than expected in real glassy systems

No P(q) measure from dynamic stochastic stability approach so far.



Maggi et al, PRB 8



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Israeloff, Nature Phys 6, 135 (2010)

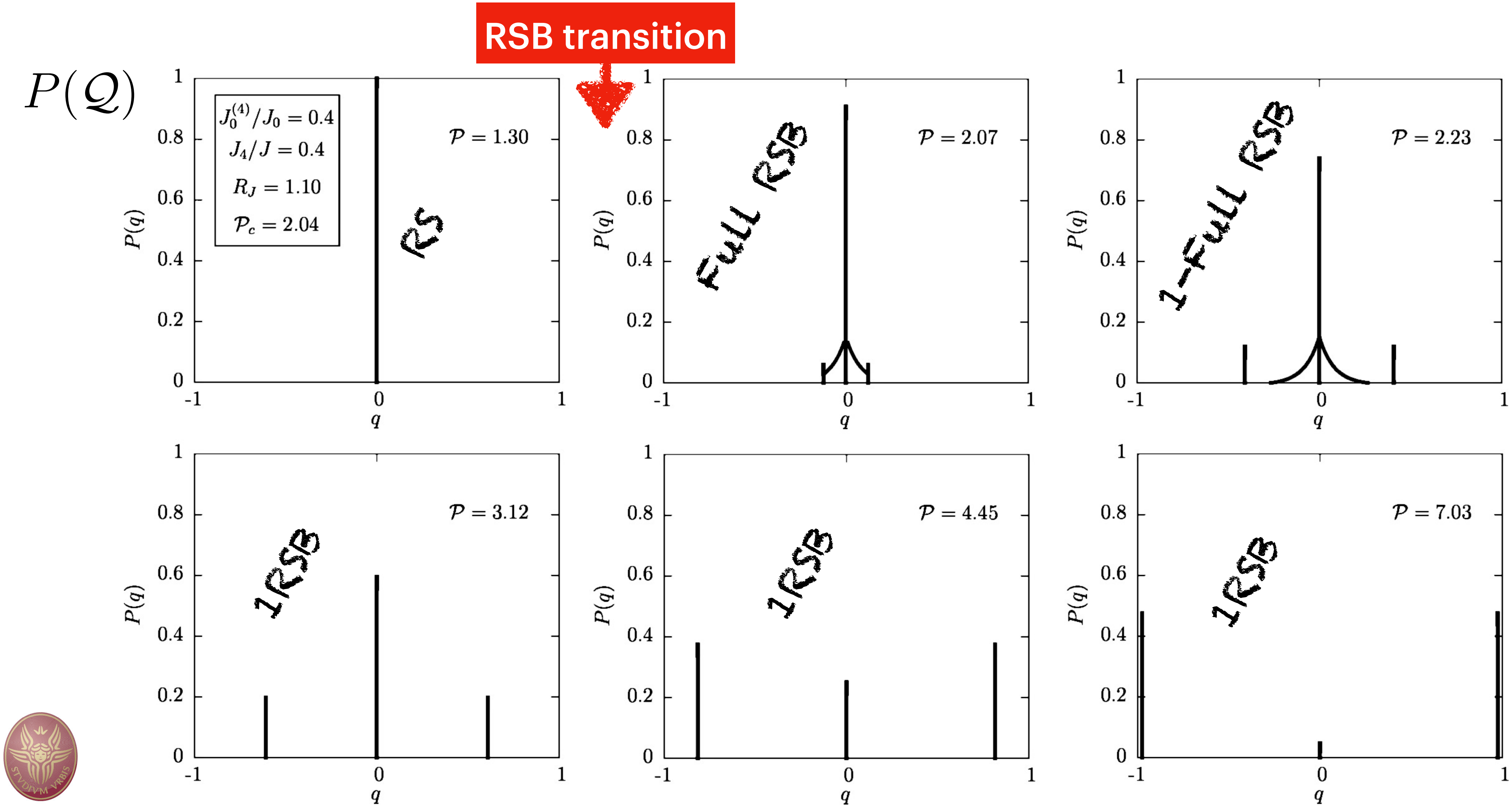
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

Intensity Fluctuation Overlap probability distribution in the Fully connected 2+4 spherical phasor model

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{N}\sum_{k=1}^{N}\left(|a_k^{(\alpha)}|^2 - \langle|a_k|^2\rangle\right)\left(|a_k^{(\beta)}|^2 - \langle|a_k|^2\rangle\right)$$

Mean-field fully connected theory $\mathcal{Q}_{ab} = q_{ab}^2$ -

F. Antenucci, A. Crisanti, LL, SciRep 2015



- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

Intensity Fluctuation Overlap - IFO
Fully connected 2+4 spherical phasor model

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{N}\sum_{k=1}^{N}\left(|a_k^{(\alpha)}|^2 - \langle |a_k|^2\rangle\right)\left(|a_k^{(\beta)}|^2 - \langle |a_k|^2\rangle\right)$$

Experimentally the same sample of random optical medium can be repeatedly illuminated by an external pumping laser under exactly the same conditions if the compound is solid (scatterers do not move between different random laser dynamics) and this implies that the mode-coupling random realization will be the same in different random laser dynamic stories.

Real replicas are feasible and in this cases, in principle

The IFO can be measured in experiments

Why not directly the phasor overlap?

Phases are unknown

$$a_k(\omega;t) = A_k(\omega;t)e^{\imath\phi_k(\omega;t)} \quad ?$$

Also instantaneous intensities are not measurable

$$I_k(\omega,t) = |a_k(\omega,t)|^2$$

We can only measure intensities on data acquisition intervals, i. e., averaged over the whole dynamics.

$$I_k(\omega) = \frac{1}{\mathcal{T}}\sum_{t=t_0}^{t_0+\mathcal{T}} |a_k(\omega,t)|^2 \simeq \langle |a_k(\omega)|^2\rangle$$

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

Intensity Fluctuation Overlap - IFO
Fully connected 2+4 spherical phasor model

$$\mathcal{Q}_{\alpha\beta} = \frac{1}{N}\sum_{k=1}^{N}\left(|a_k^{(\alpha)}|^2 - \langle |a_k|^2\rangle\right)\left(|a_k^{(\beta)}|^2 - \langle |a_k|^2\rangle\right)$$

If real replicas are feasible IFO's can be measured in experiments

The experimental intensity fluctuation of the shot (replica) α with respect to the average spectrum is

$$\Delta_k^\alpha \equiv \frac{I_k^\alpha - \bar{I}_k}{\sqrt{\sum_{k=1}^N \left(I_k^\alpha - \bar{I}_k\right)^2}} \simeq \mathcal{N}\left(I_k^\alpha - \bar{I}_k\right)$$

$$\bar{I}_k \equiv \frac{1}{N_R}\sum_{\alpha=1}^{N_R} I_k^\alpha$$

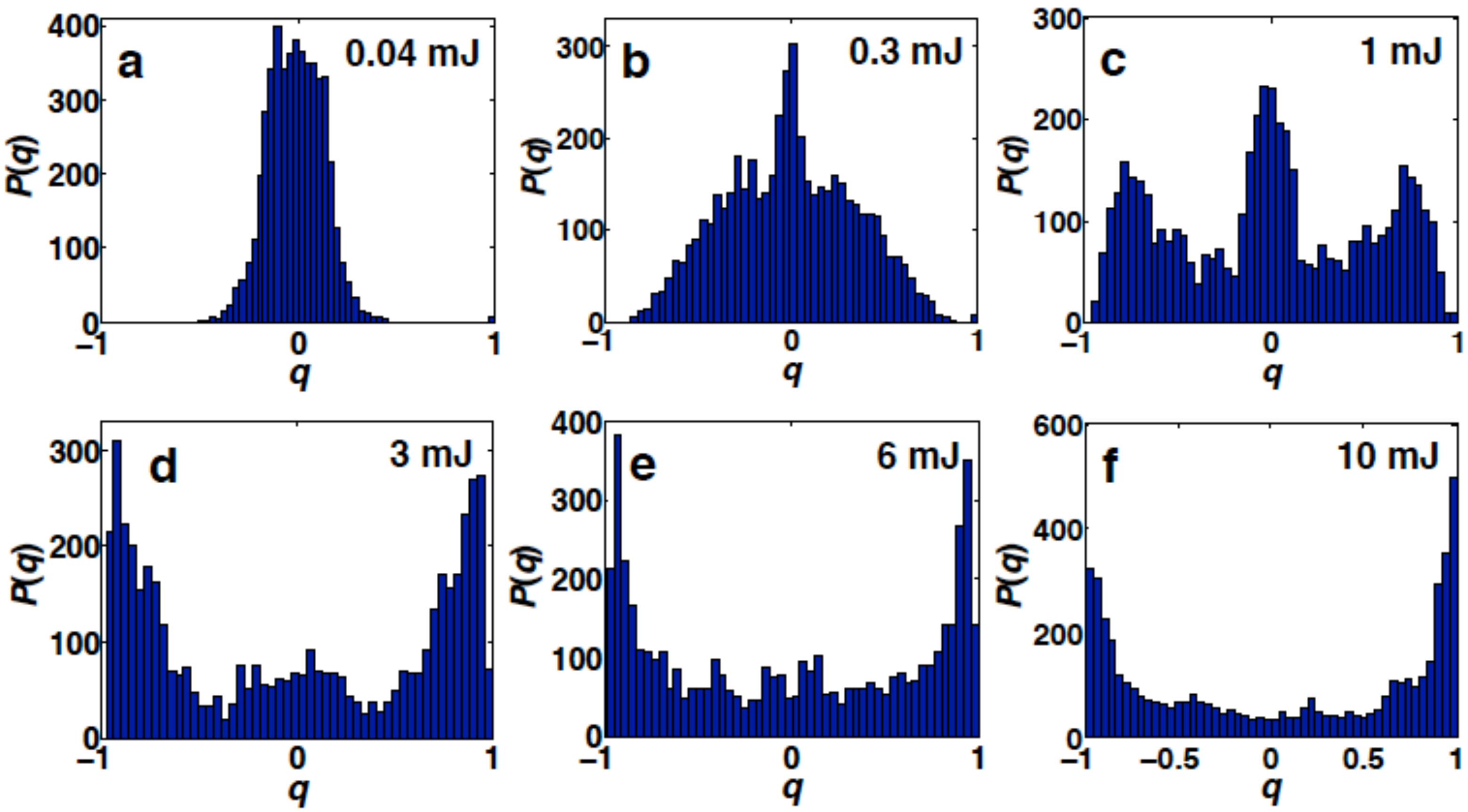
and their overlap is

$$\mathcal{Q}_{\alpha\beta} \equiv \frac{1}{N}\sum_{k=1}^{N}\Delta_k^\alpha\Delta_k^\beta$$

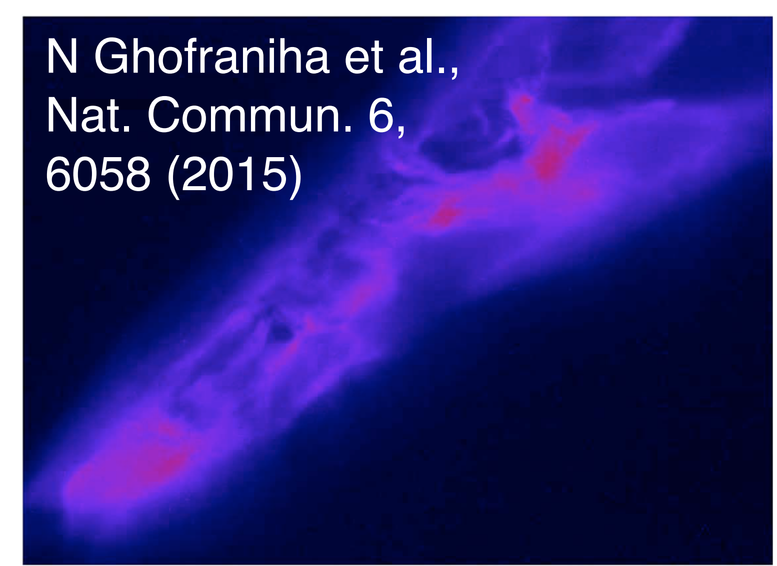
These overlaps can be measured over many shots at fixed external pumping power and the procedure can be repeated at different external pumping powers across the lasing transition to see how their distribution behaves

- Intermezzo: the experimental measurement of the Parisi distribution of overlaps

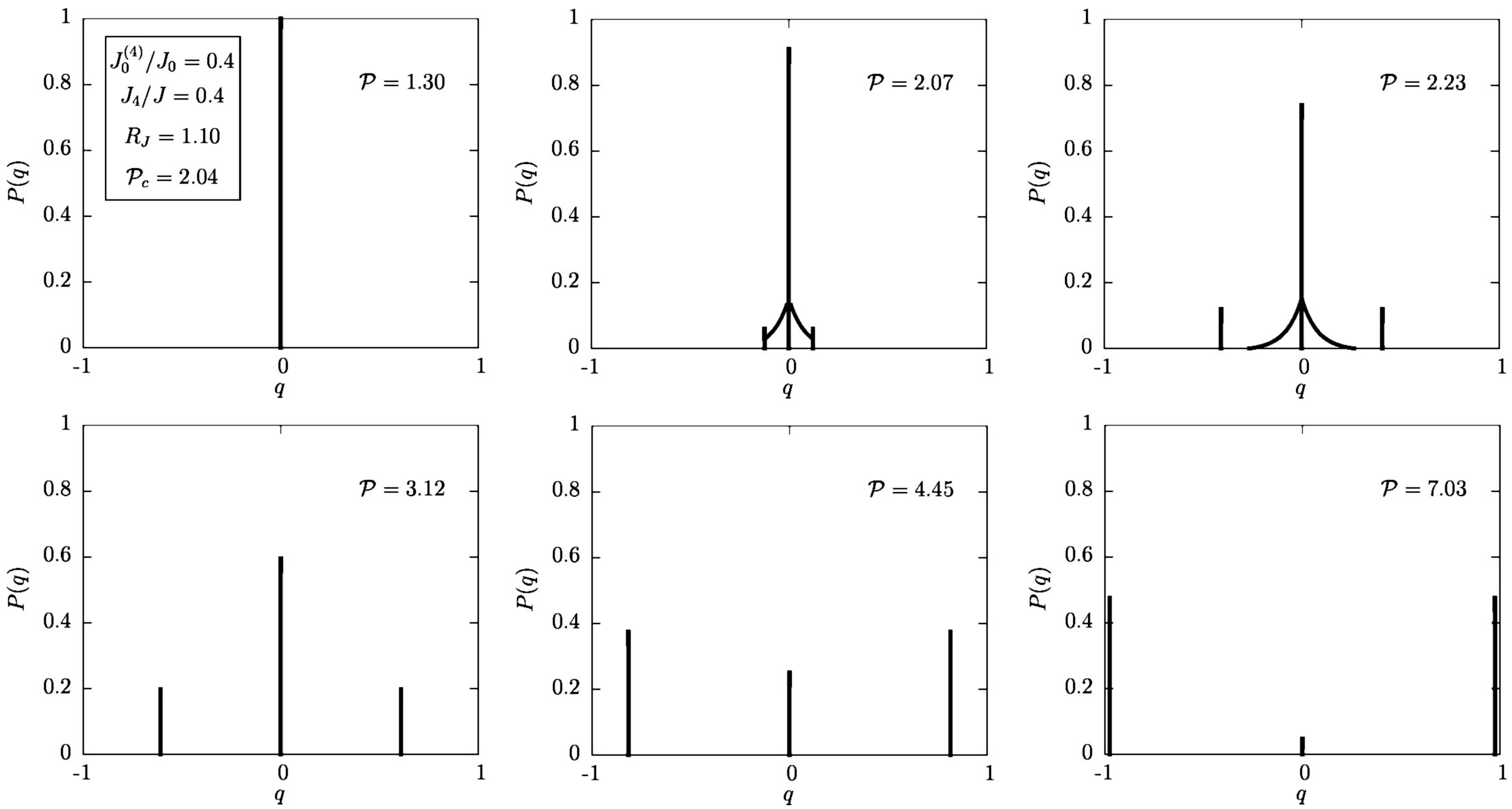
Replica Symmetry Breaking is experimentally detected in IFO in the T5OCx Random Laser



T5OCx grains



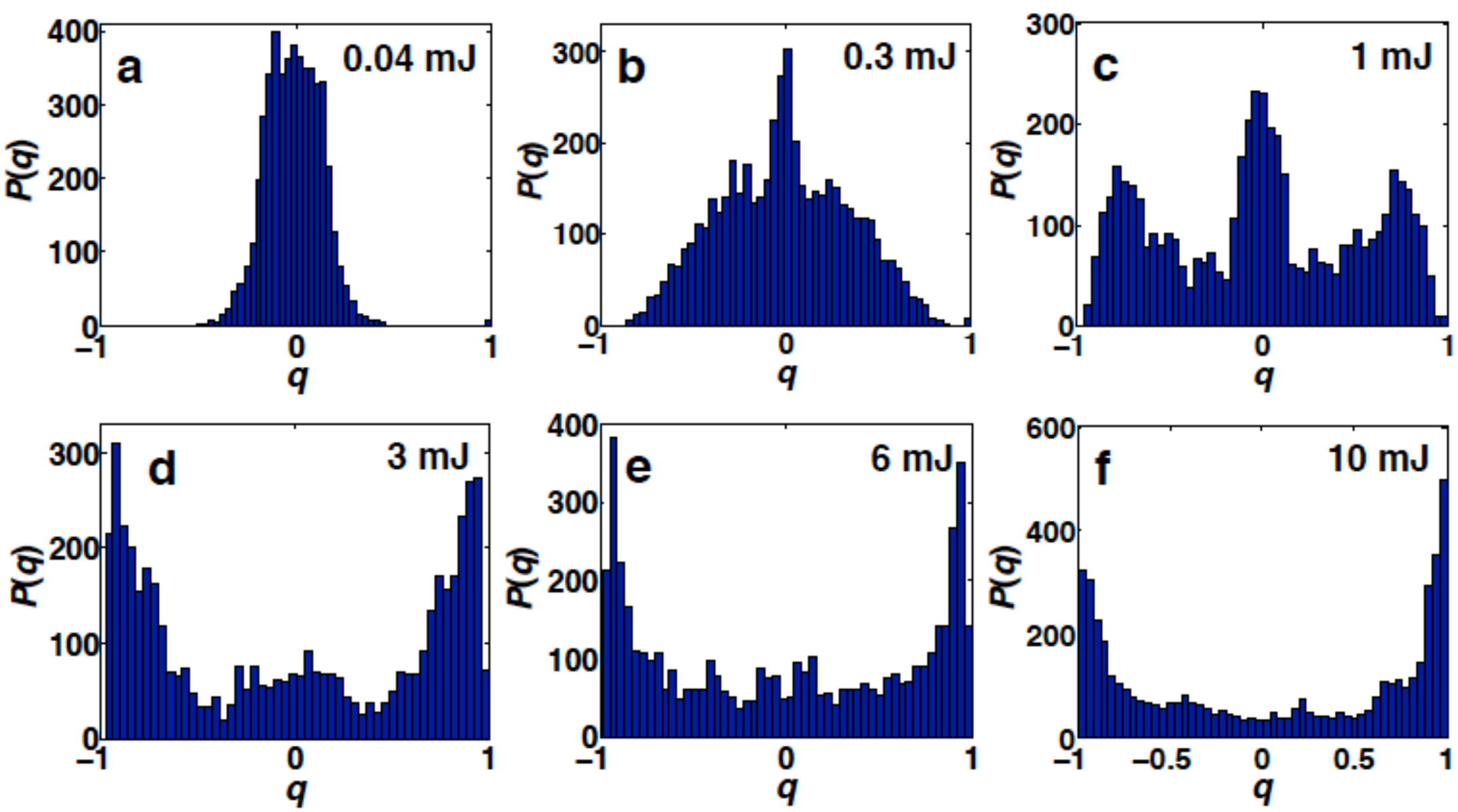
• Intermezzo: the experimental measurement of the Parisi distribution of overlaps



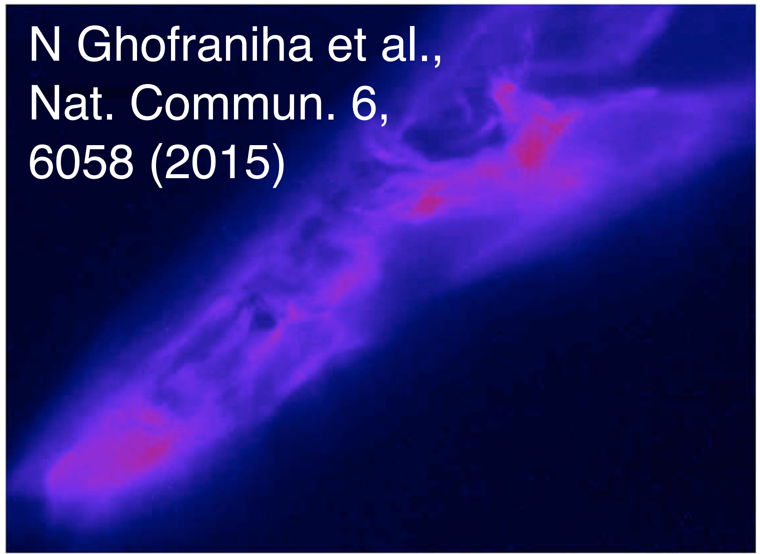
Mean-field $\mathcal{Q}_{ab} = q_{ab}^2$
fully connected theory

F. Antenucci, A. Crisanti, LL, SciRep 2015

Element-element relation: any Replica Symmetry Breaking is detectable and it is experimentally detected in IFO in the T5OCx Random Laser
first direct measurement



T5OCx grains



- In between theory and experiment

Differences between mean-field theory on complete graph and experiments

Fully connected RSB theory

- In the narrowband approximation **mode-locking does not play any role** and the **interaction graph is complete**
- In the *thermodynamic limit* of **infinite** number of modes
- under constant energy and **effective equilibrium** assumptions and using an equilibrium ensemble of *instantaneous* modes

$$I_k(t) = |a_k(t)|^2$$

In experiments:

- **mode-locking** is expected to occur and the interaction **graph is unknown**, as well as the magnitude of the couplings
- the number of modes is **finite** and also their resolution
- **equilibration is not under control** and we do not have access to instantaneous resonances but only to the total intensity acquired

$$\langle I_k(t)\rangle = \frac{1}{\mathcal{T}}\int_{t_0}^{t_0+\mathcal{T}} dt\ I_k(t)$$

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- In between theory and experiment: a mode-locking model

We include mode frequencies and frequency matching for a more realistic model

We also simplify some (momentarily less essential) features:
* only 4-"spins" (no losses, flat gain profile)
* comb-like distributed frequencies (easier combinatorics)

spherical 4-phasor mode-locked random laser

$$\mathcal{H}[\boldsymbol{a}] = -\sum_{\boldsymbol{k}|\mathrm{FMC}(\boldsymbol{k})} J_{k_1 k_2 k_3 k_4} \overline{a}_{k_1} a_{k_2} \overline{a}_{k_3} a_{k_4} + \mathrm{c.c.}$$

$$\mathrm{FMC}(\boldsymbol{k}) : |\omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_{k_4}| \lesssim \gamma$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

Random couplings

$$\mathcal{P}(J_{k_1 \cdots k_p}) = \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left\{-\frac{J^2_{k_1 \cdots k_p}}{2\sigma_p^2}\right\}$$

$$\sigma_p^2 = \overline{J^2_{k_1 \ldots k_p}} \propto \frac{1}{N^{p-2}}$$

$$p = 4$$

$$\sigma_4^2 = \overline{J^2_{k_1 k_2 k_3 k_4}} \propto \frac{1}{N^2}$$

Equispaced comb-like frequencies

$$\omega_k = \omega_0 + \delta\omega \ k \longrightarrow |k_1 - k_2 + k_3 - k_4| = 0 \quad , k_i = 1, \ldots, N$$

G. Gradenigo, F. Antenucci, LL, Phys Rev Res 2020
J. Niedda, G. Gradenigo, LL and G. Parisi, SciPost Phys. 2023
J. Niedda, G. Gradenigo, LL, JSTAT 2023

In between theory and experiment: a mode-locking model

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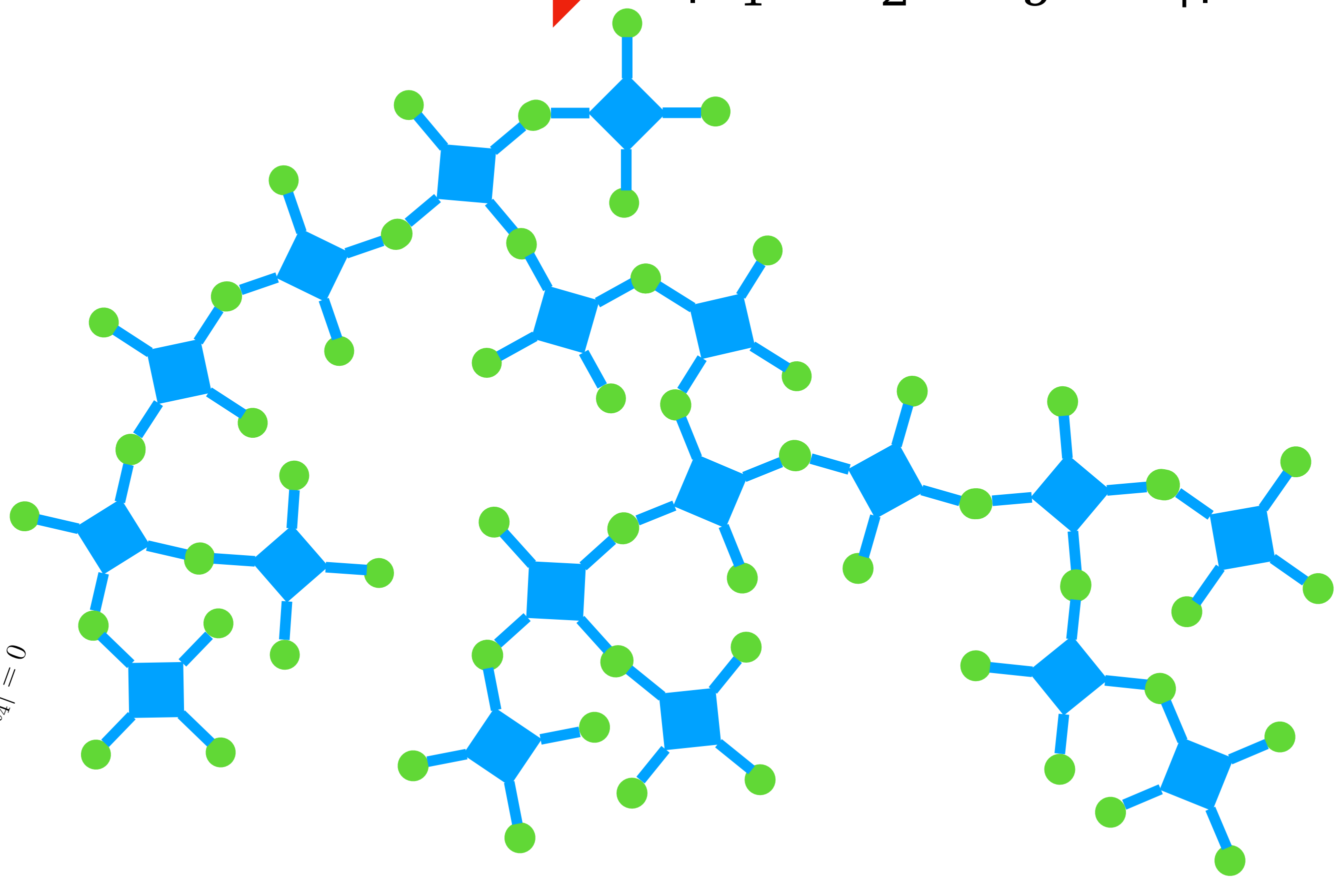
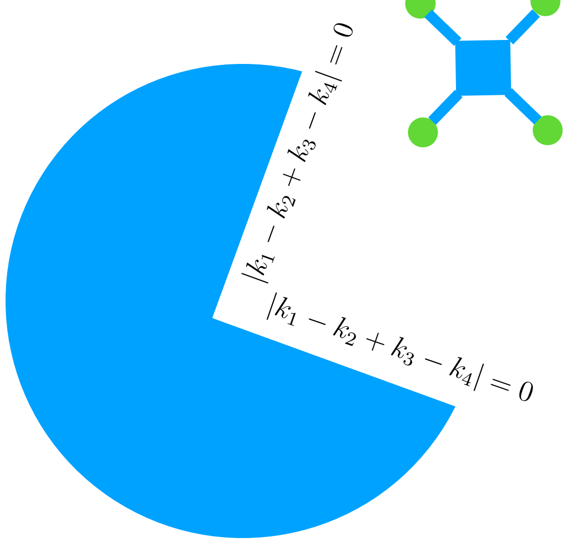
Effect of frequency matching condition on interaction graph connectivity

$$|k_1 - k_2 + k_3 - k_4| = 0$$





spherical 4-phasor mode-locked random laser

A Marruzzo, P Tyagi, F Antenucci, A Pagnani, LL, Sci. Rep. 7, 3463 (2017)

Effect of frequency matching condition on interaction graph connectivity

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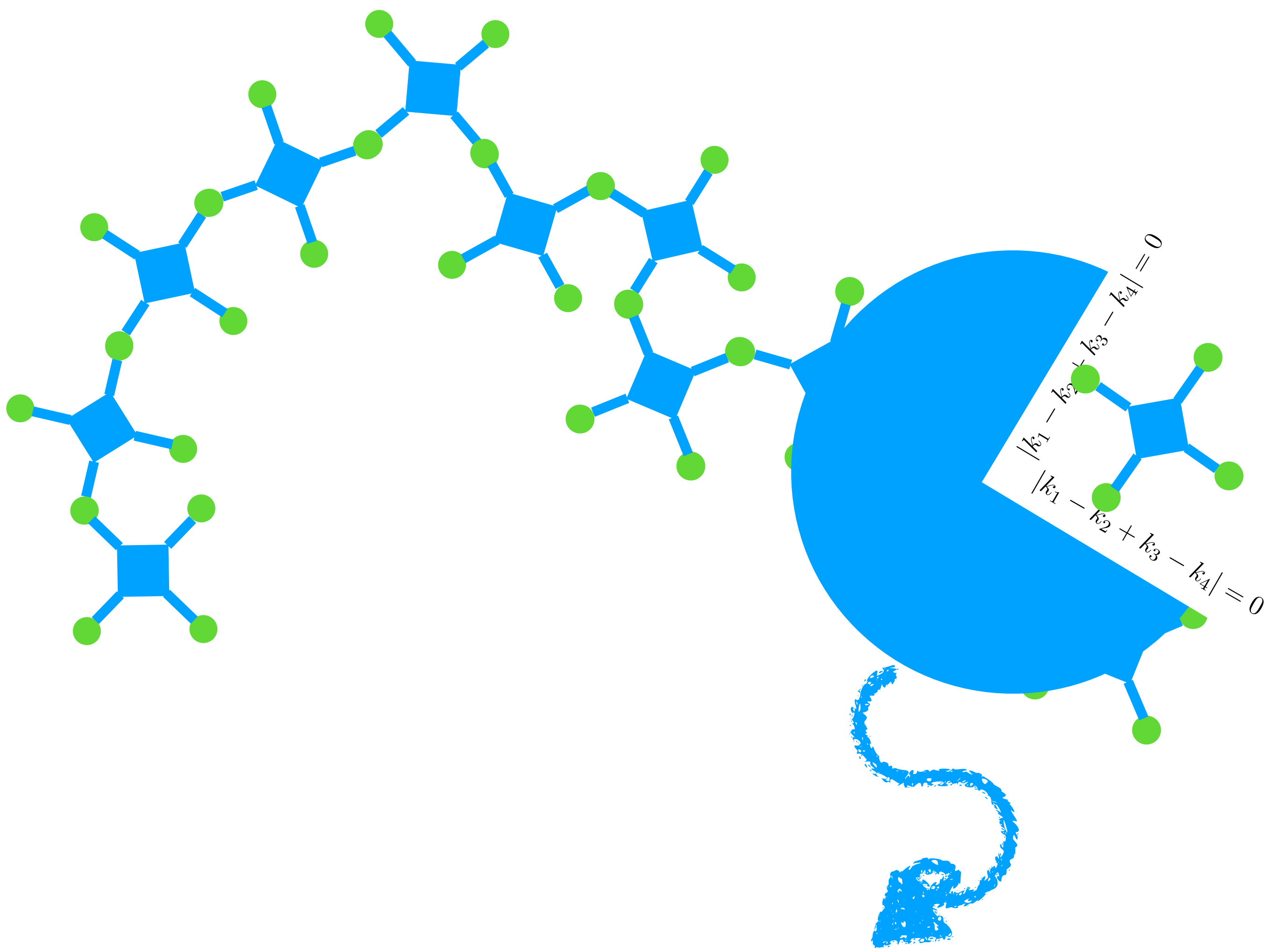


spherical 4-phasor mode-locked random laser

$$\binom{N}{4} \times \left(\frac{2}{3N} + \frac{1}{3N^3}\right) = \mathcal{O}(N^3)$$

Complete factor graph

Frequency matching condition ~~decimation~~ pruning

A Marruzzo, P Tyagi, F Antenucci, A Pagnani, LL, Sci. Rep. 7, 3463 (2017)

Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

* Exchange Monte Carlo for equilibrium study

* Parallel computation of contribution to single mode energy update

Energy of a configuration of N phasors

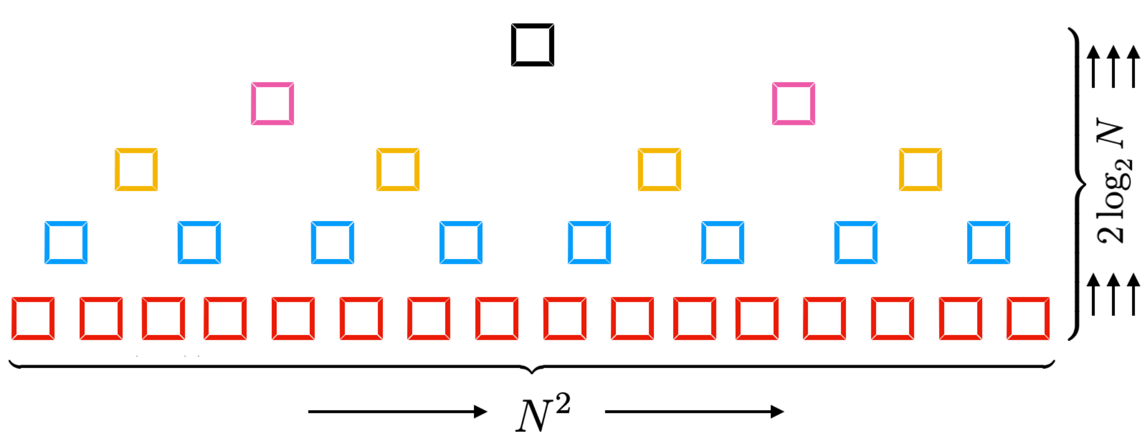
$$\mathcal{H}[\boldsymbol{a}] = -\sum_{\boldsymbol{k}|\mathrm{FMC}(\boldsymbol{k})} J_{k_1k_2k_3k_4}\bar{a}_{k_1}a_{k_2}\bar{a}_{k_3}a_{k_4} + \mathrm{c.c.} = \underbrace{J_{1234}\bar{a}_1a_2\bar{a}_3a_4 + J_{1235}\bar{a}_1a_2\bar{a}_3a_5 + J_{1239}\bar{a}_1a_2\bar{a}_3a_9 + \ldots}$$

is the sum of $\underbrace{\begin{pmatrix} N \\ 4 \end{pmatrix}}_{\text{Complete factor graph}} \times \underbrace{\left(\frac{2}{3N} + \frac{1}{3N^3}\right)}_{\text{Frequency matching condition pruning}} = \mathcal{O}(N^3)$ terms

Single mode energy update

$a_k \to a'_k \Rightarrow \Delta E \equiv \mathcal{H}[\boldsymbol{a}'] - \mathcal{H}[\boldsymbol{a}]$ is the sum of $\mathcal{O}(N^2)$ terms

Each term in the sum computed in parallel on GPU kernels

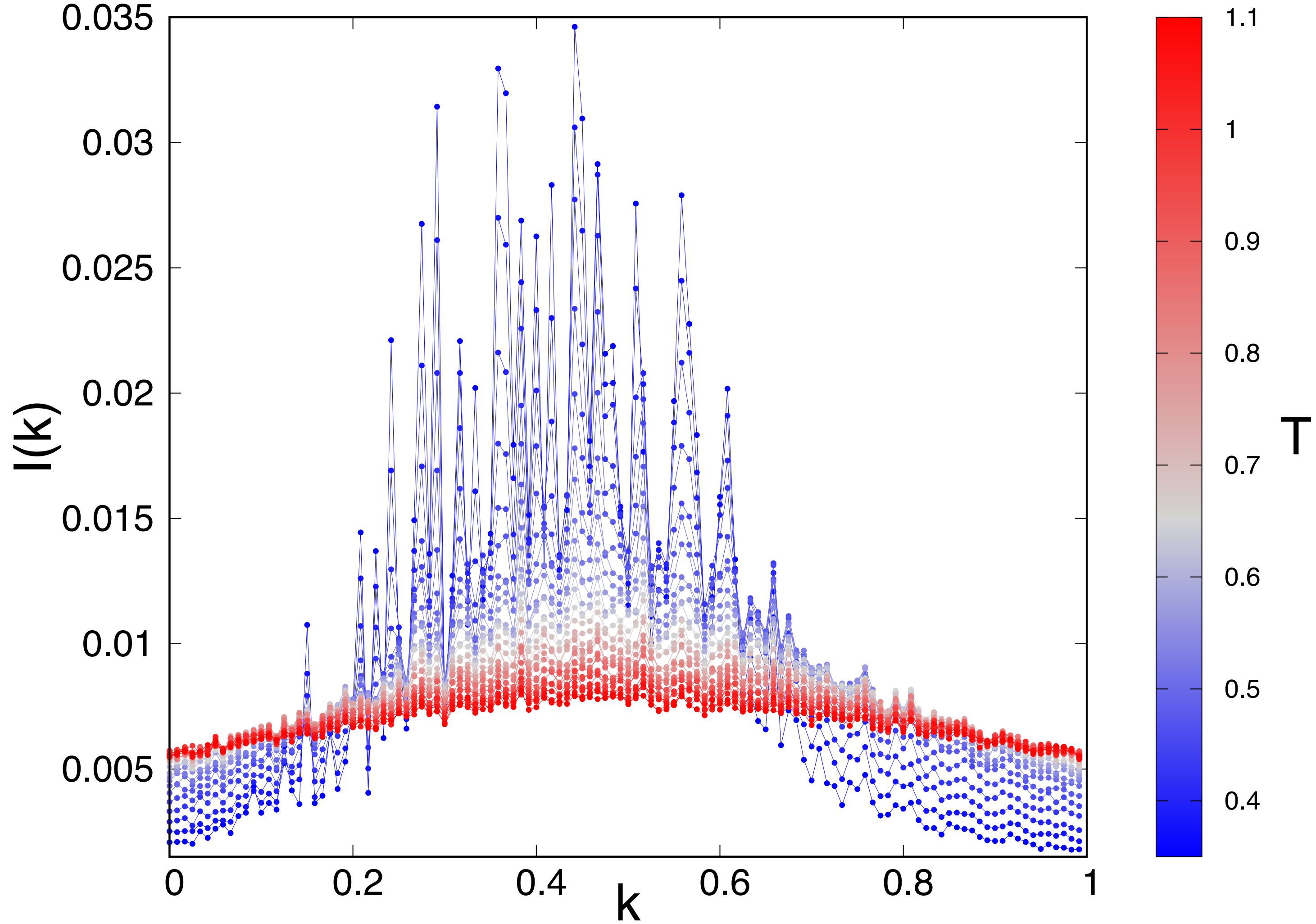


The time required to carry out the sum scales as $\log_2 N$

Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

Spectral emission

Spectral narrowing and sharp resonances as pumping increases



$$\mathcal{P}^2 = \frac{\epsilon^2}{T} = \frac{1}{T_{\text{photonic}}} \qquad |k_1 - k_2 + k_3 - k_4| = 0$$

Hui Cao et. al , Appl. Phys. Lett. 1998

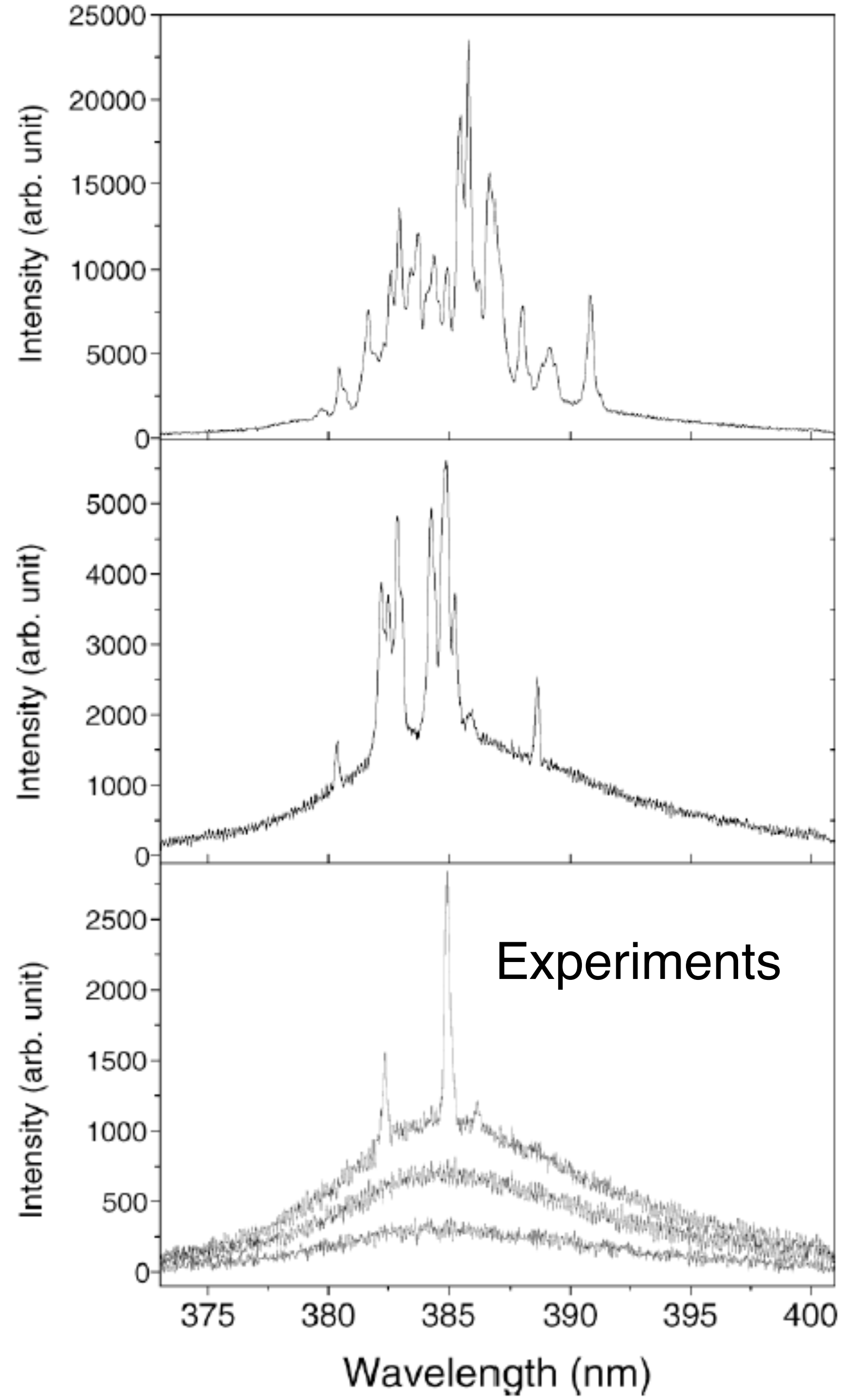


Figure 1. Spectra of emission from ZnO powder when the excitation intensity is (from bottom to top) 400, 562, 763, 875 and 1387 kW cm^{-2}.

J. Niedda, G. Gradenigo, LL and G. Parisi, SciPost Phys. 2023

Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

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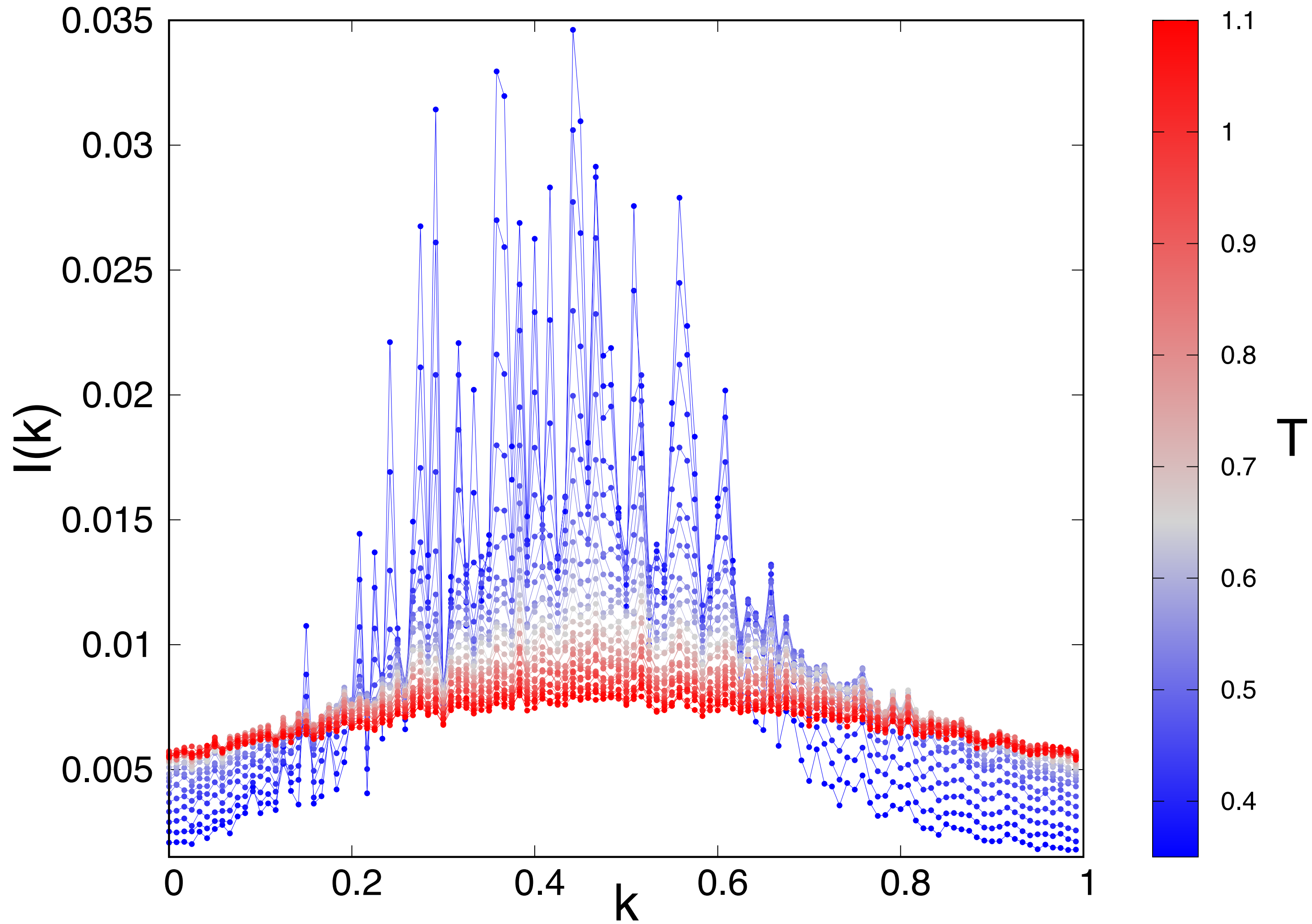

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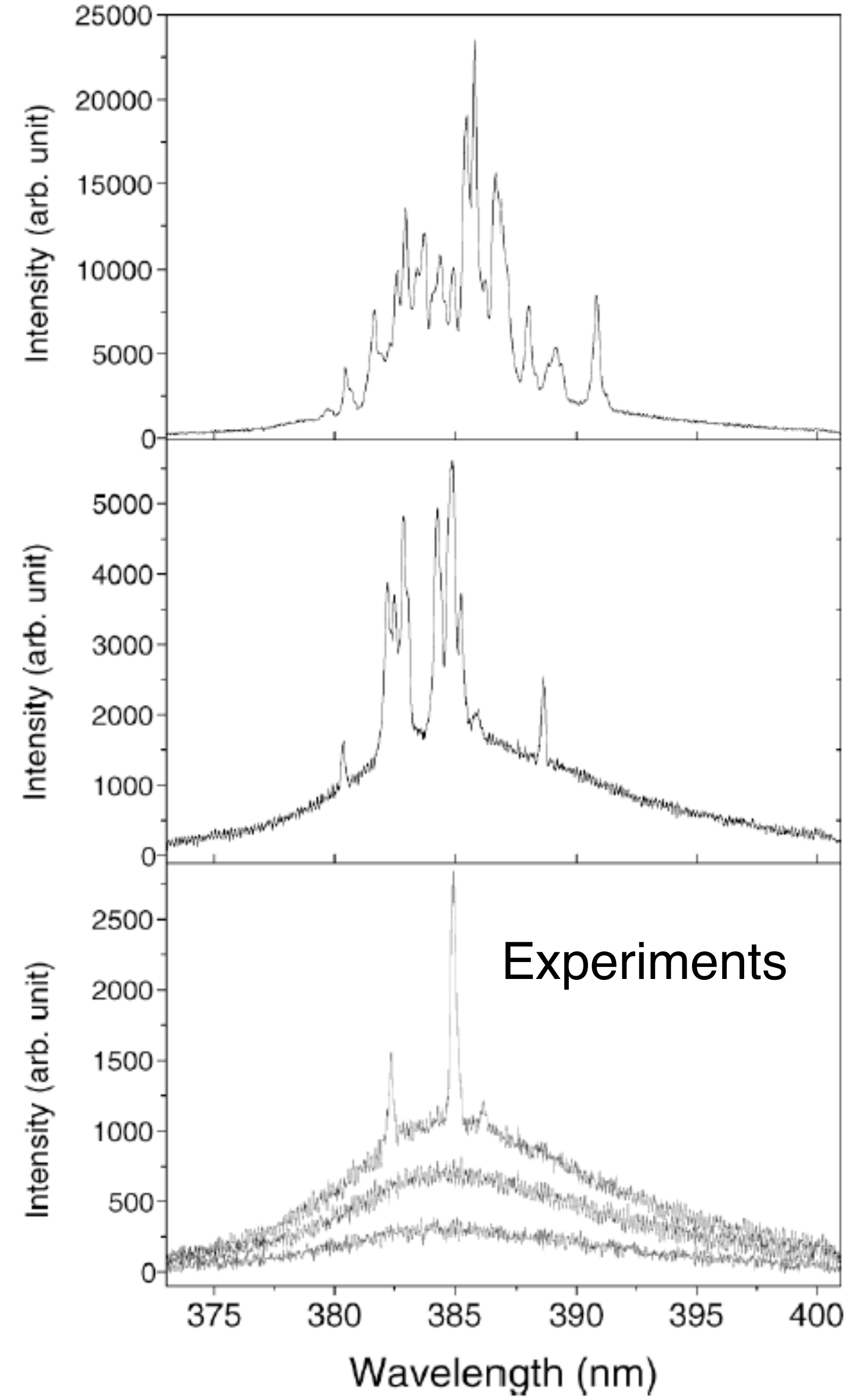


Figure 1. Spectra of emission from ZnO powder when the excitation intensity is (from bottom to top) 400, 562, 763, 875 and 1387 kW cm^{-2}.

$$\mathcal{P}^2 = \frac{\epsilon^2}{T} = \frac{1}{T_{\text{photonic}}}$$

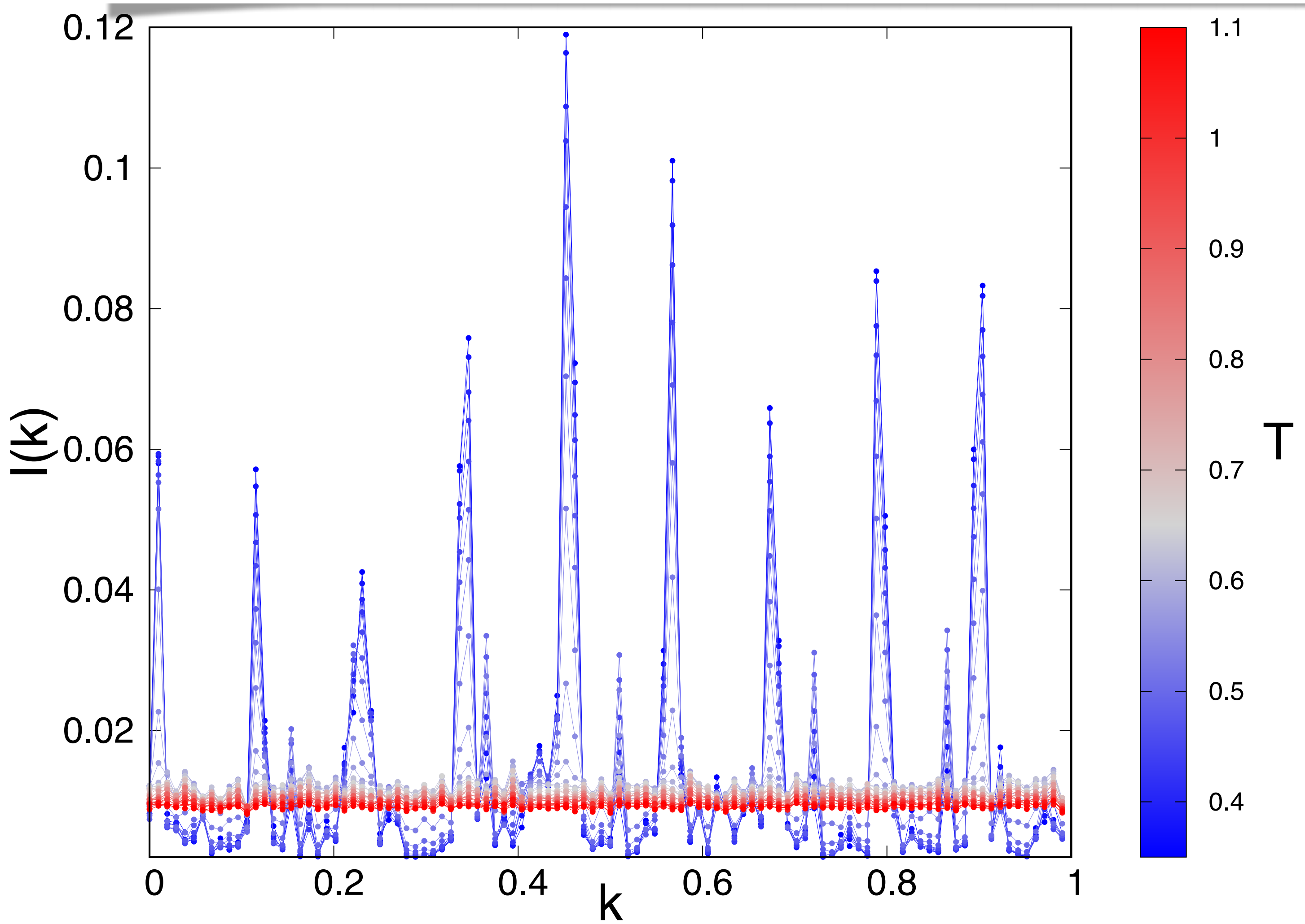
$$|k_1 - k_2 + k_3 - k_4| = 0$$

Strong finite size effects (large sizes of continuous variables on densely connected interaction graphs take long times to simulate)

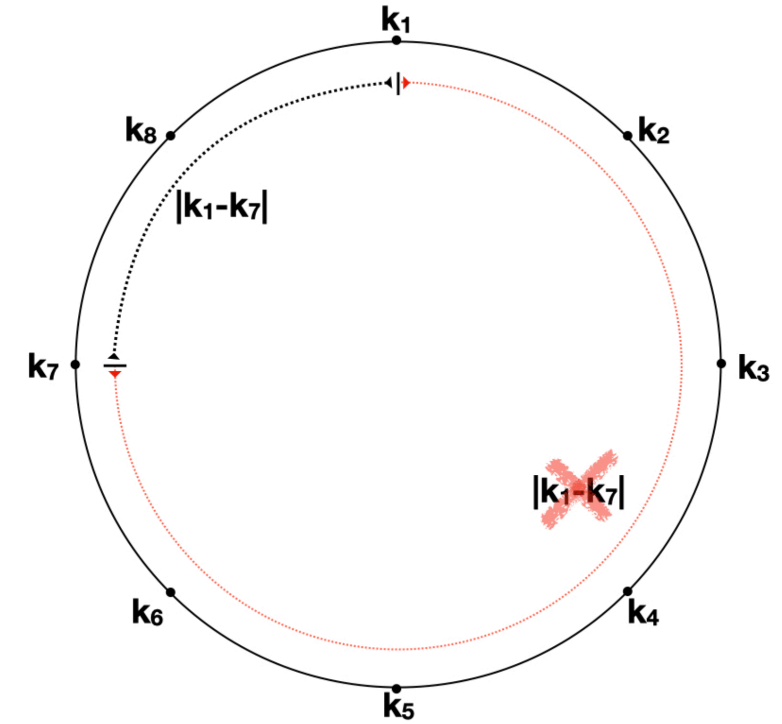
J. Niedda, G. Gradenigo, LL and G. Parisi, SciPost Phys. 2023

Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

A trick to reduce large FSE is to impose periodic boundary conditions *on the mode frequencies*



Frequency periodic boundary conditions

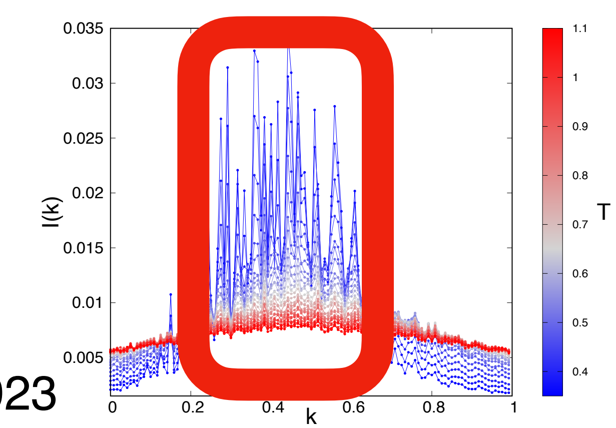


$$|k_1 - k_2 + k_3 - k_4| = 0$$

$$\mathcal{P}^2 = \frac{\epsilon^2}{T} = \frac{1}{T_{\text{photonic}}}$$

As if only central modes count

$$|k_a - k_b| = \begin{cases} |k_a - k_b| & \text{if } |k_a - k_b| \le \left[\frac{N}{2}\right] \\ N - |k_a - k_b| & \text{if } |k_a - k_b| \ge \left[\frac{N}{2}\right] \end{cases}$$



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Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

Phase transition and universality class

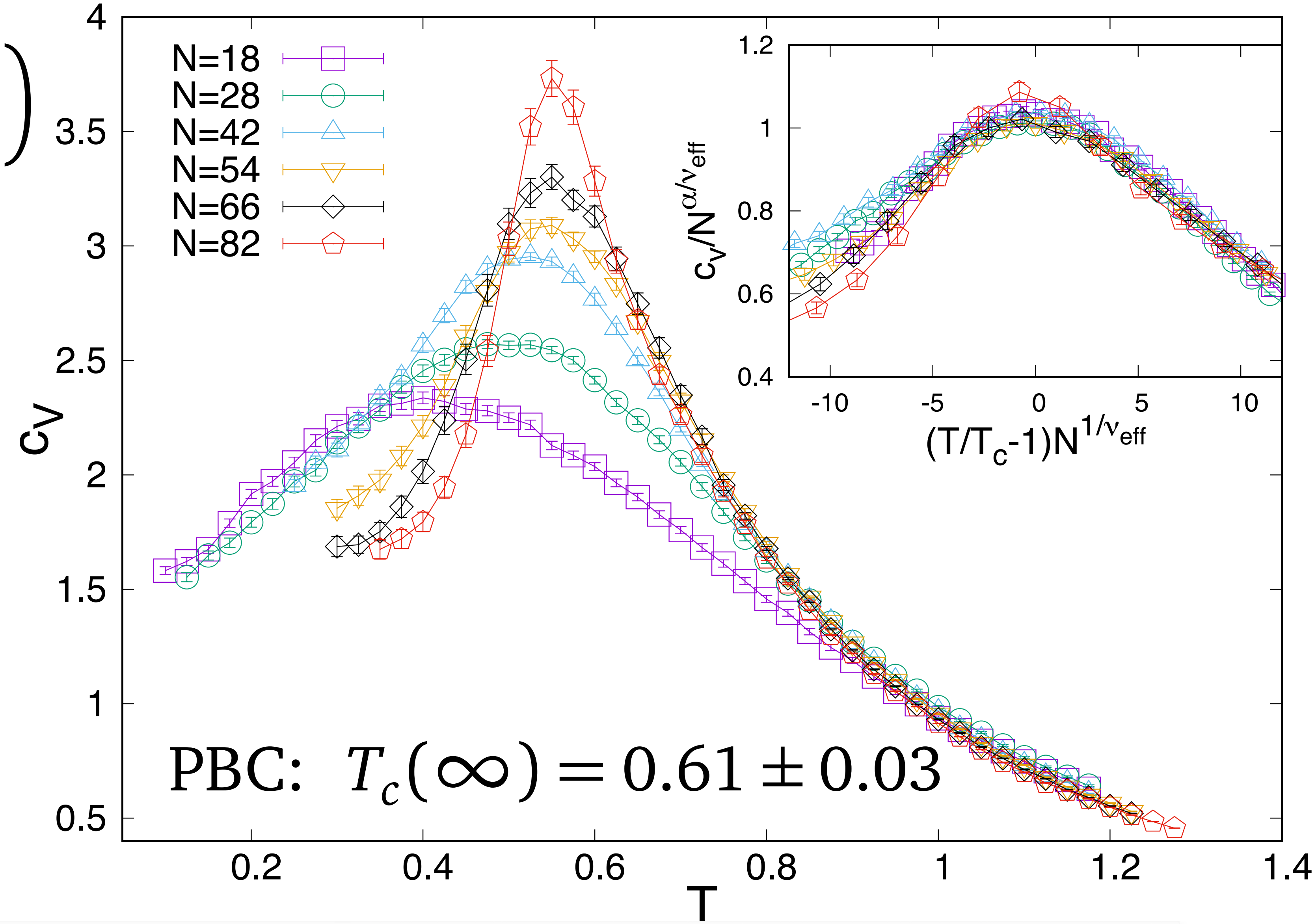
$$c_{V_N}(T) = N^{\frac{\alpha}{\nu_{\rm eff}}} \hat{f}_{C_{V_N}}\left(N^{\frac{1}{\nu_{\rm eff}}}\ t_N\right)$$

$$\nu_{\rm eff} = 2\beta + \gamma$$

$$1 \leq \nu_{\rm eff} \leq 2$$

for a mean-field model

PBC: Periodic boundary conditions on frequenciesto reduce finite size effects



Strong finite size effects, yet **compatible with a mean-field theory**

PBC: $\alpha = 0.27 \pm 0.05$, $1/\nu_{\rm eff} = 0.86 \pm 0.14$

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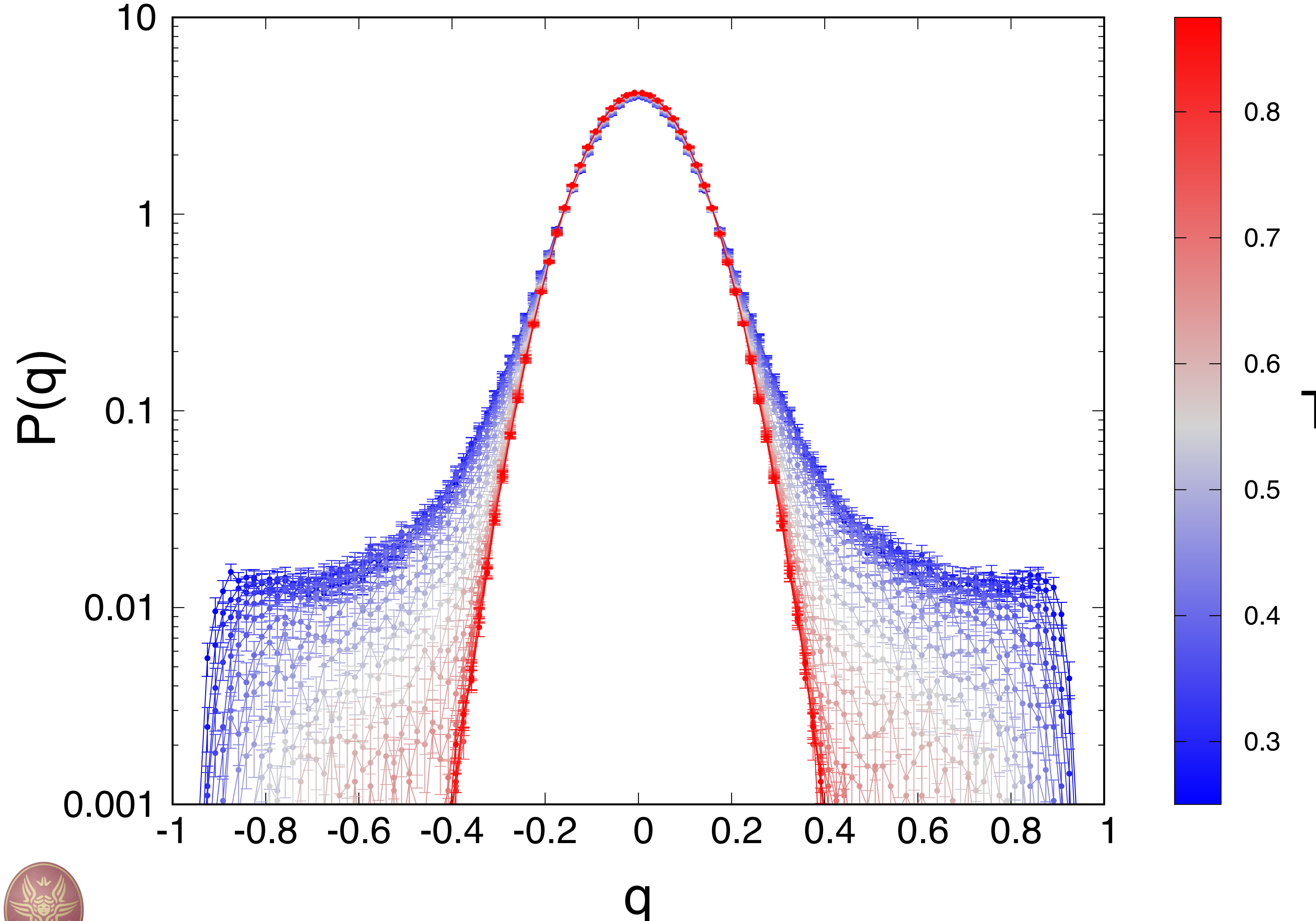
Monte Carlo simulations of the spherical 4-phasor mode-locked random laser

Replica symmetry breaking

non-trivial distribution below the transition point:
non-gaussian tails and side peaks at low T



PBC: $T_c(\infty) = 0.61 \pm 0.03$

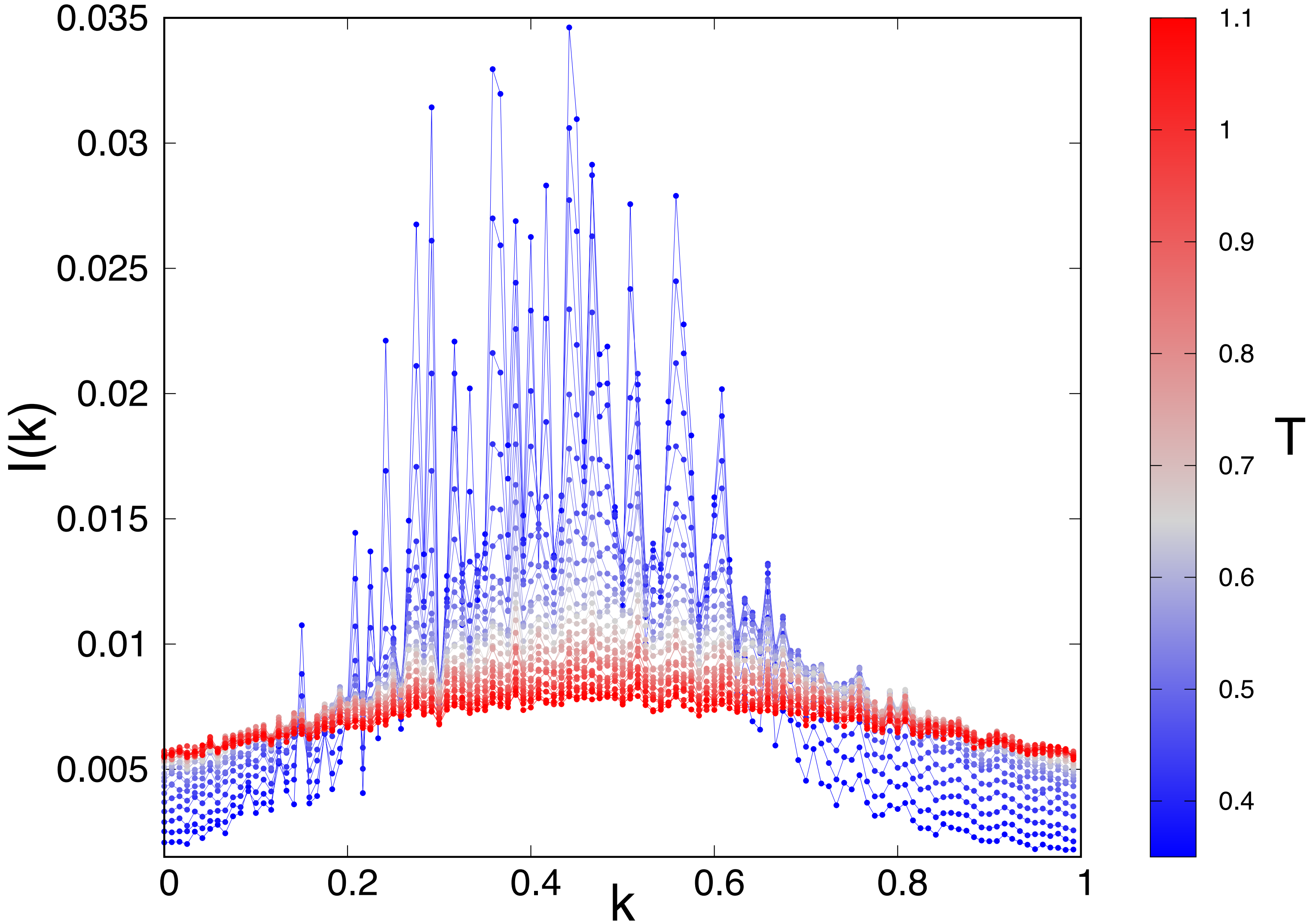
Periodic boundary conditions on frequencies

Outline

- Standard and random lasers
- Statistical physics approach to laser physics
  Theory for ultrafast mode-locked multimode lasers (order, closed cavity)
  Theory for random lasers: a mode-locked spin-glass theory (disorder, open cavity)
- The narrow-band solution, phase diagrams, replica symmetry breaking, a new overlap: intensity fluctuation overlap
- Intermezzo: the experimental measurement of the Parisi distribution of overlaps
- In between theory and experiment: a mode-locking model
  Monte Carlo dynamics simulation with exchange Monte Carlo, GPU parallel computing
- Power distribution among modes in the glassy light regime: condensation vs equipartition at high pumping
- Outlook (work in progress)

Monte Carlo simulations of the spherical 4-phasor ML RL

How is the power distributed among modes?



$$I_k = |a_k|^2$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

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Monte Carlo simulations of the spherical 4-phasor ML RL

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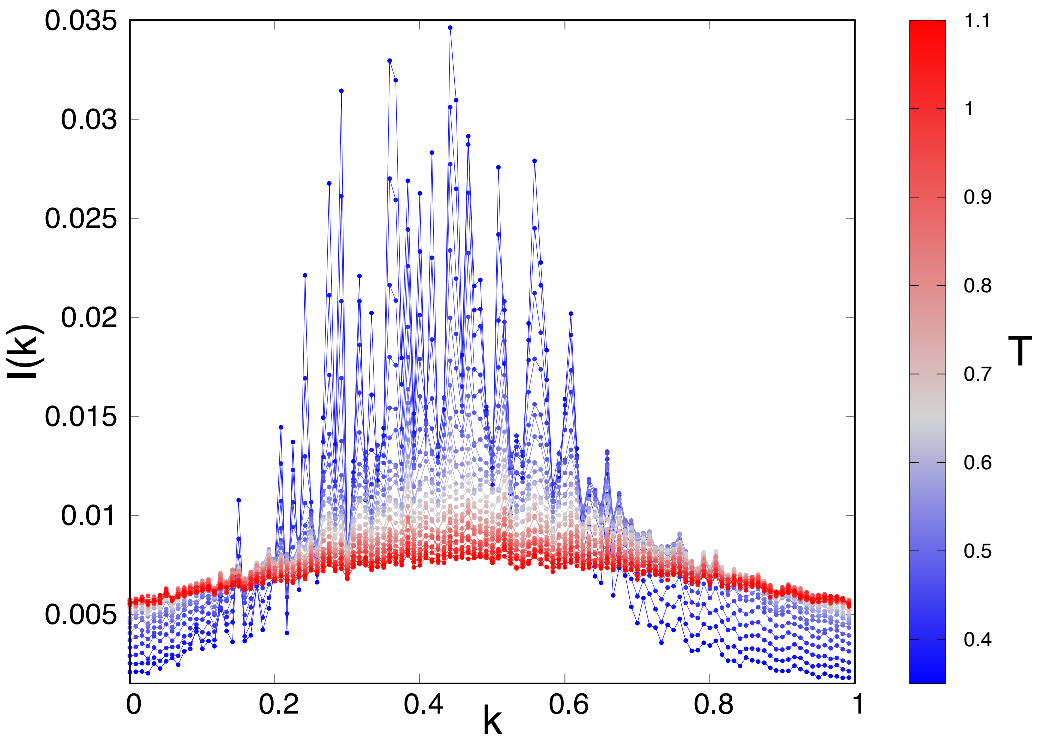
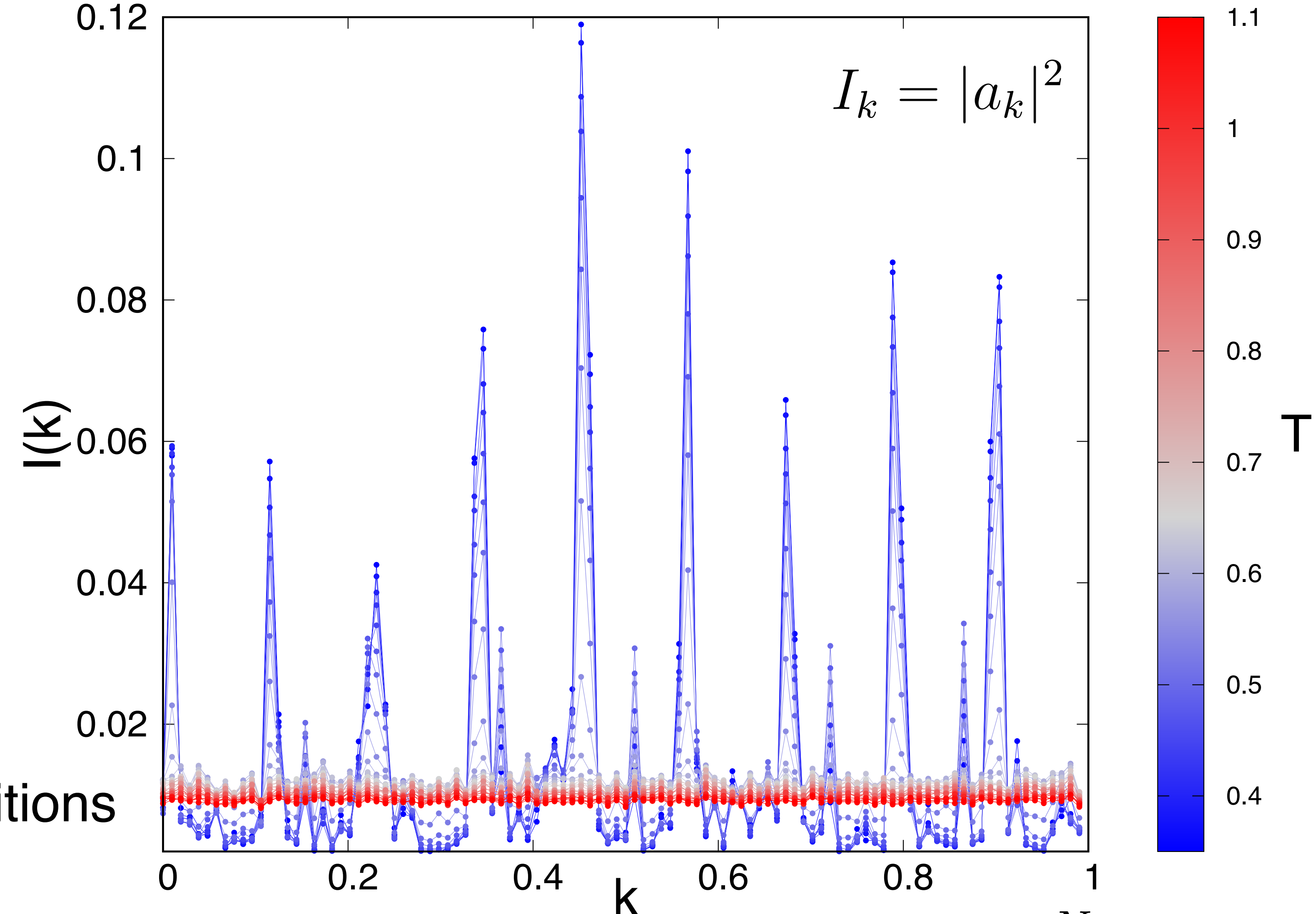


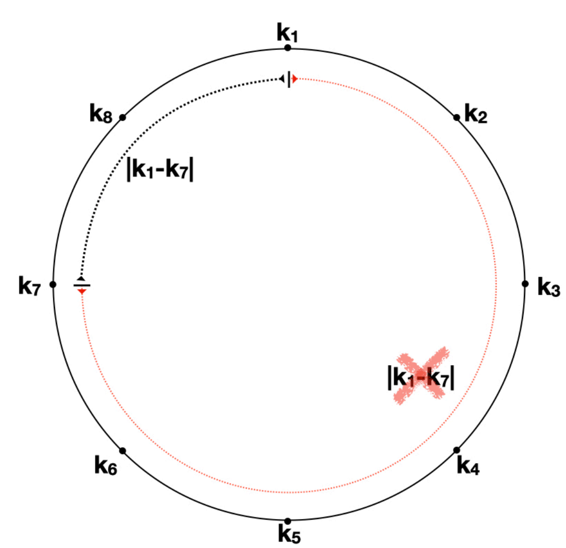


Frequency Periodic Boundary Conditions

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

$$|k_1 - k_2 + k_3 - k_4| = 0$$

$$|k_a - k_b| = \begin{cases} |k_a - k_b| & \text{if } |k_a - k_b| \leq \left[\frac{N}{2}\right] \\ N - |k_a - k_b| & \text{if } |k_a - k_b| \geq \left[\frac{N}{2}\right] \end{cases}$$



PBC reduce
finite size effects

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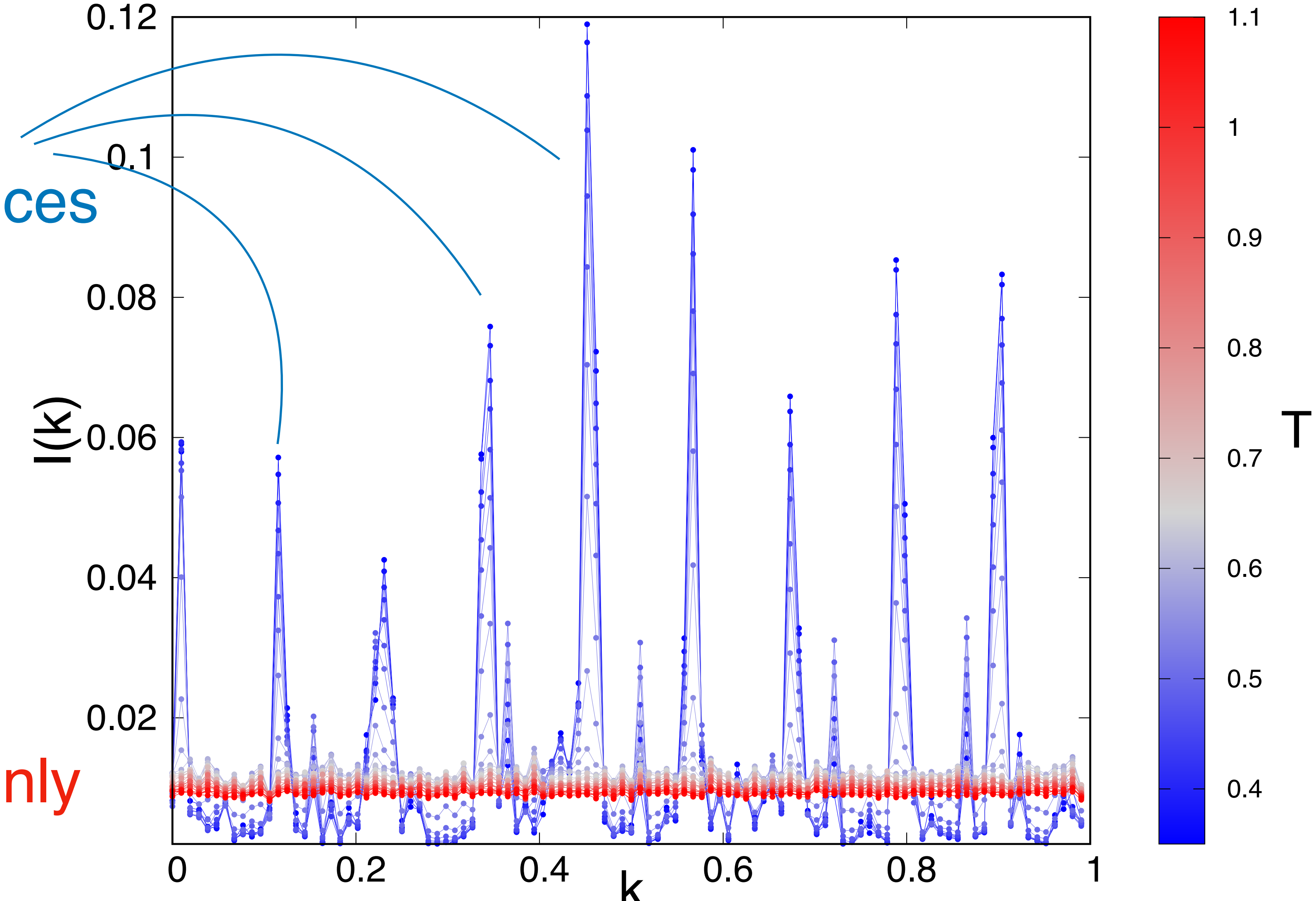
How is the power distributed among modes?

High P/low T: intensity is concentrated on a few resonances



High T/low P: intensity is evenly distributed among modes

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

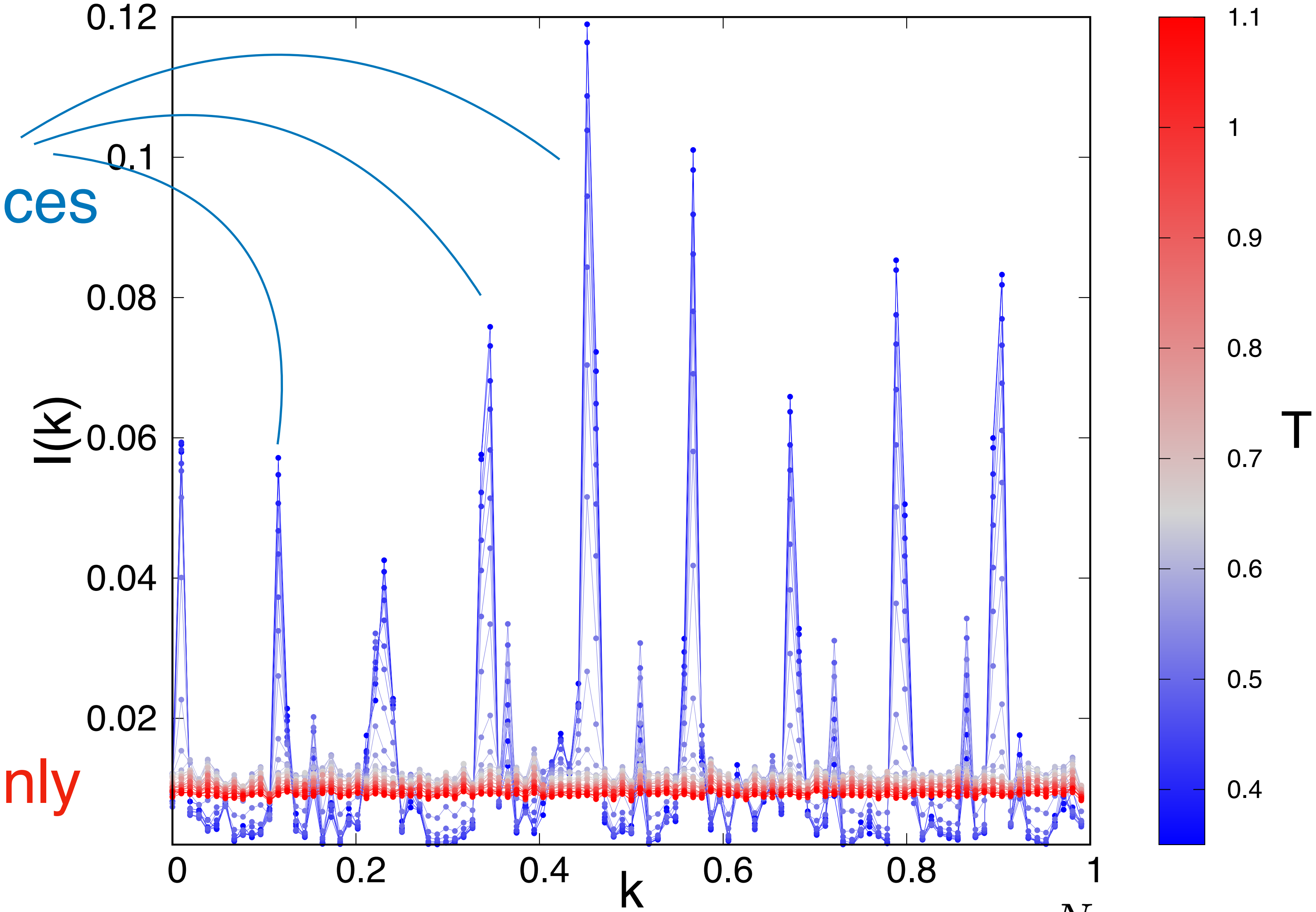
Monte Carlo simulations of the spherical 4-phasor ML RL

How is the power distributed among modes?

High P/low T: intensity is concentrated on a few resonances

Are “a few” very few?
Do they take all the intensity distributed in the system?

High T/low P: intensity is evenly distributed among modes



$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

$$\sum_{k=1}^{N} \frac{|a_k|^2}{\epsilon} = N$$

Monte Carlo simulations of the spherical 4-phasor ML RL

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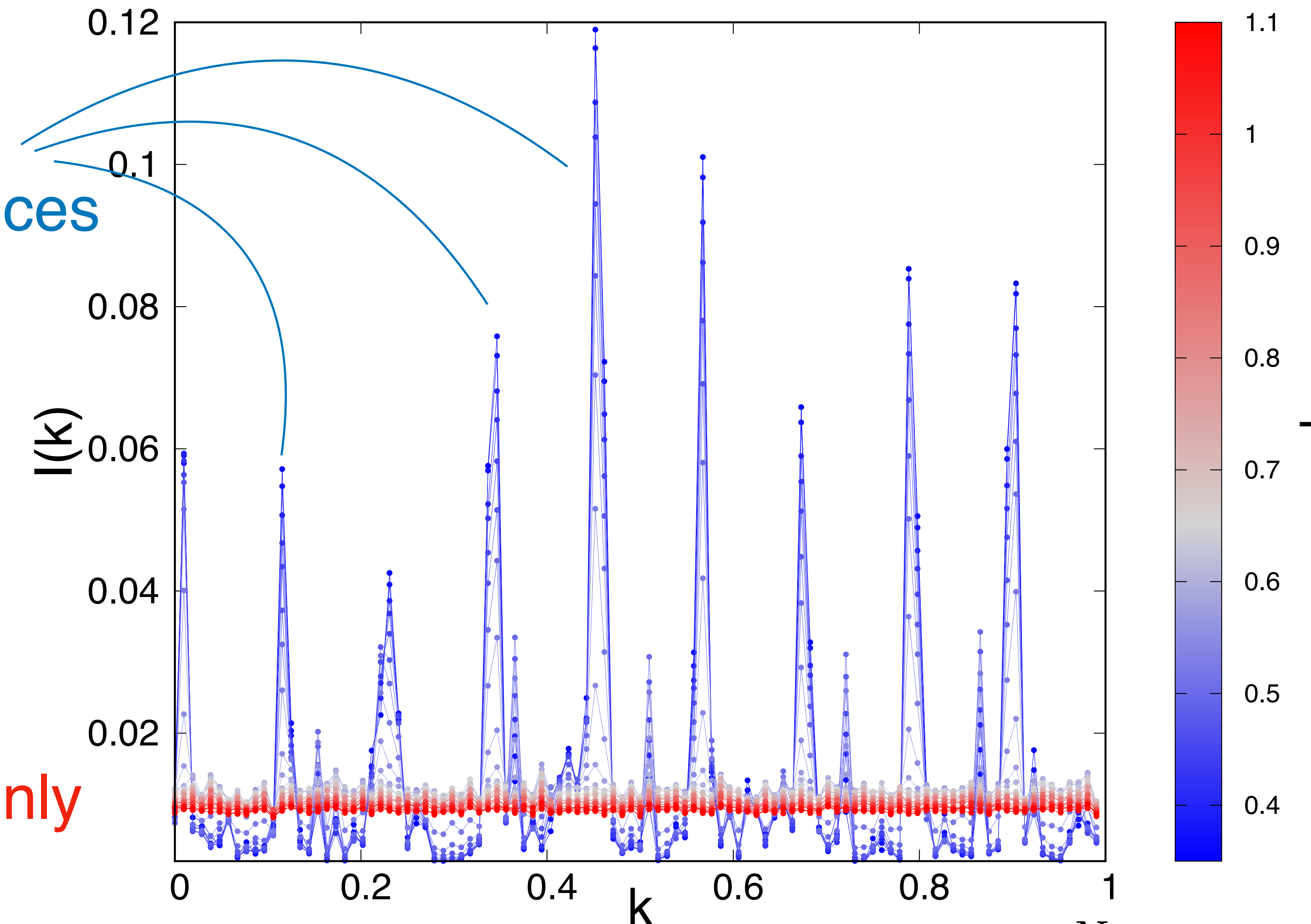

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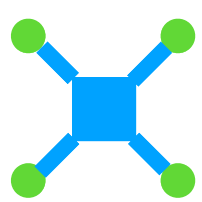

$\{\boldsymbol{a}\}$: CONDENSED $\quad |a_k| \in \square \propto \sqrt{N} \quad , \quad |a_k| \not\in \square = 0$

?

$$\sum_{k=1}^{N} \frac{|a_k|^2}{\epsilon} = N$$

$\{\boldsymbol{a}\}$: UNIFORM $\quad |a_k| \simeq 1 \ \forall\ k = 1, \ldots, N$

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How is the power distributed among modes?

Participation ratio: $$Y_2 = \left\langle \frac{\sum_{k=1}^N I_k^2}{(\sum_{k=1}^N I_k)^2} \right\rangle = \frac{1}{N^2} \left\langle \sum_{k=1}^N I_k^2 \right\rangle \qquad I_k = |a_k|^2$$

vs T

$$\lim_{N\to\infty} Y_2 = \text{const}$$

condensation

$$Y_2 \sim \frac{1}{N}$$

equipartition

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

Monte Carlo simulations of the spherical 4-phasor ML RL

How is the power distributed among modes?

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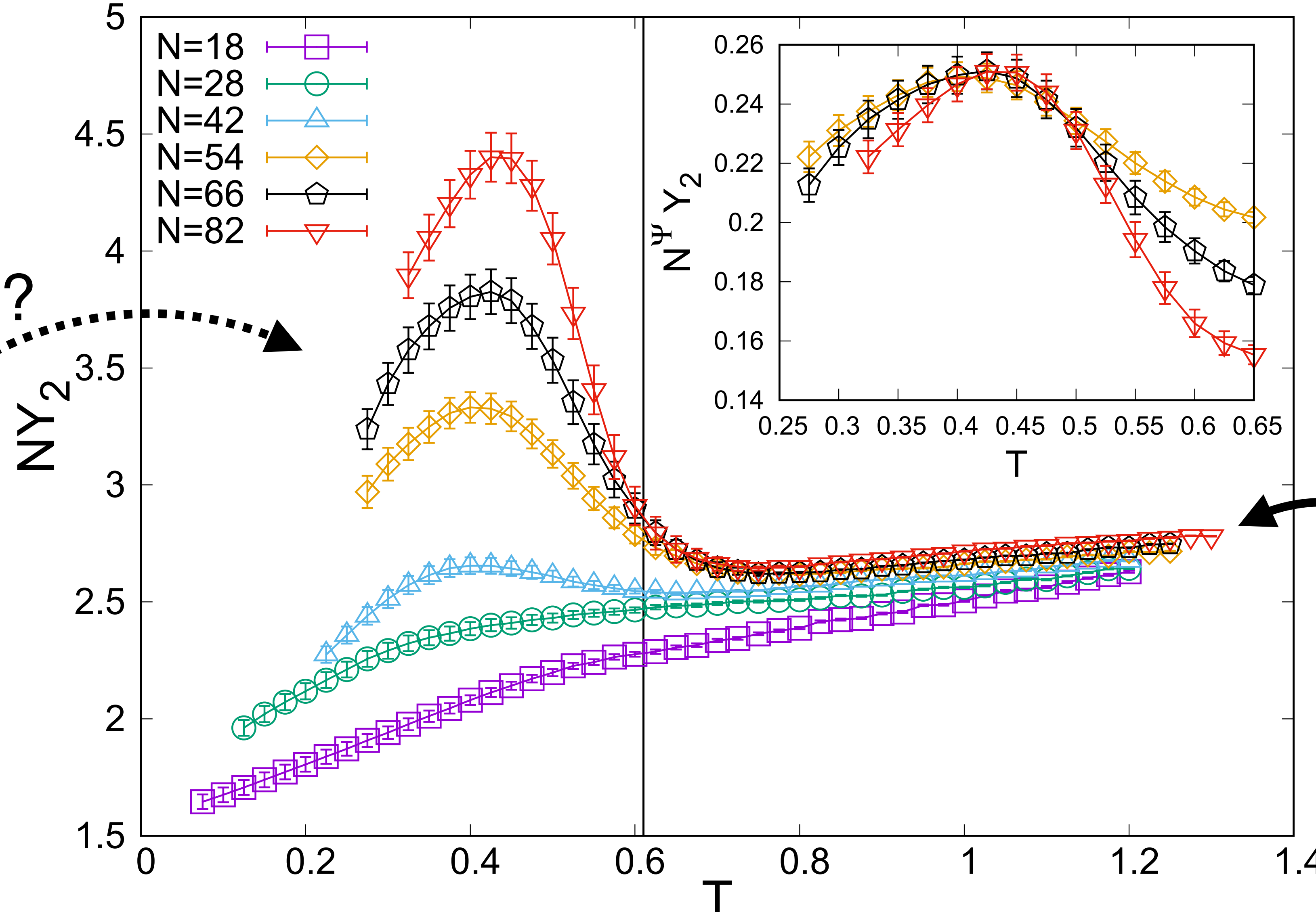
condensation ?

$\times N$



$\times N$

$$\lim_{N\to\infty} Y_2 = \text{const}$$

condensation

$$Y_2 \sim \frac{1}{N}$$

equipartition

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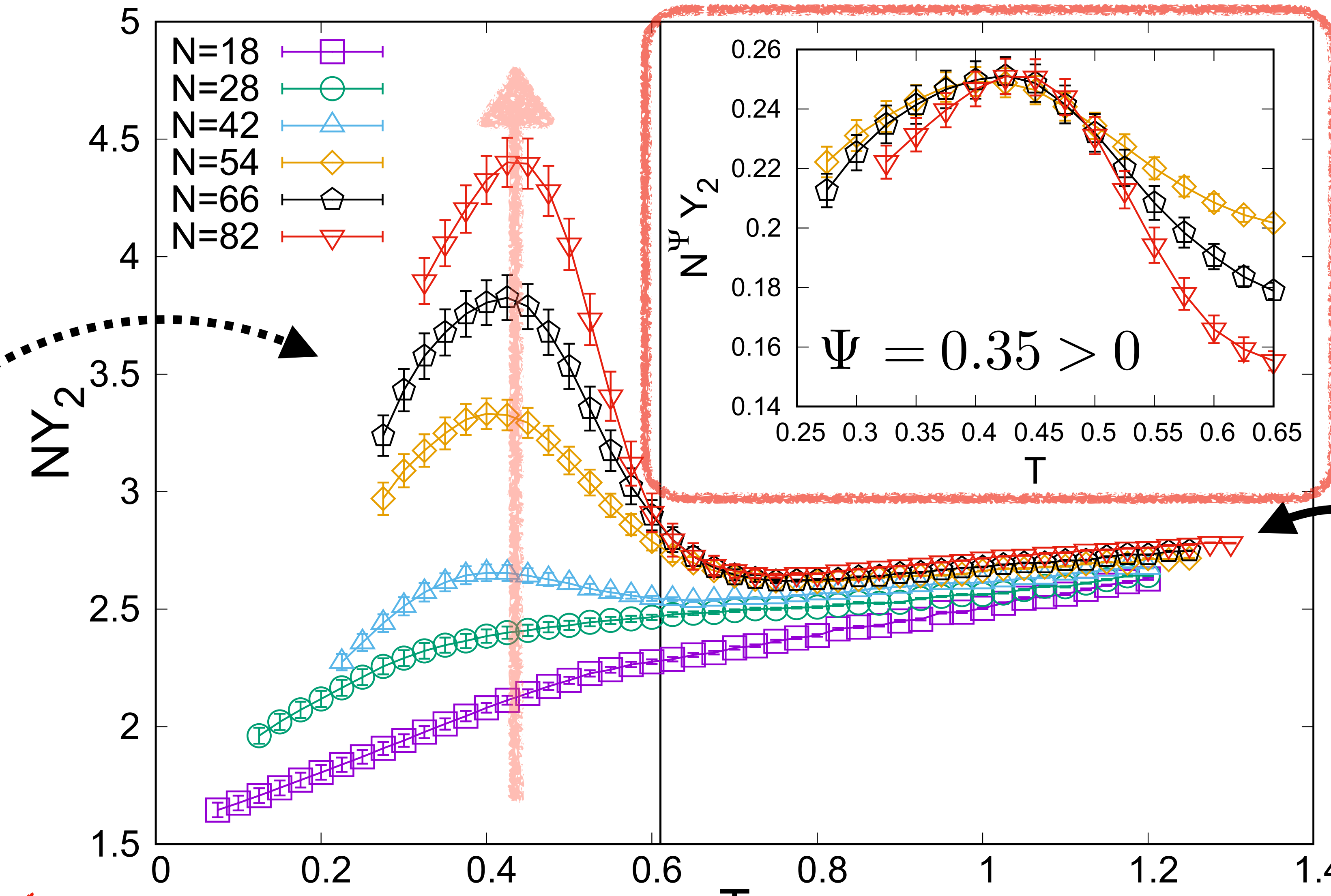
Monte Carlo simulations of the spherical 4-phasor ML RL

Power pseudo-condensation

Participation ratio: $$Y_2 = \left\langle \frac{\sum_{k=1}^{N} I_k^2}{(\sum_{k=1}^{N} I_k)^2} \right\rangle = \frac{1}{N^2}\left\langle \sum_{k=1}^{N} I_k^2 \right\rangle \qquad I_k = |a_k|^2$$



$$\cancel{\lim_{N\to\infty} Y_2 = \mathrm{const}}$$

~~condensation~~

no condensation at low T
(but no equipartition either):
pseudo-condensation

$$Y_2 \sim \frac{1}{N}$$

equipartition

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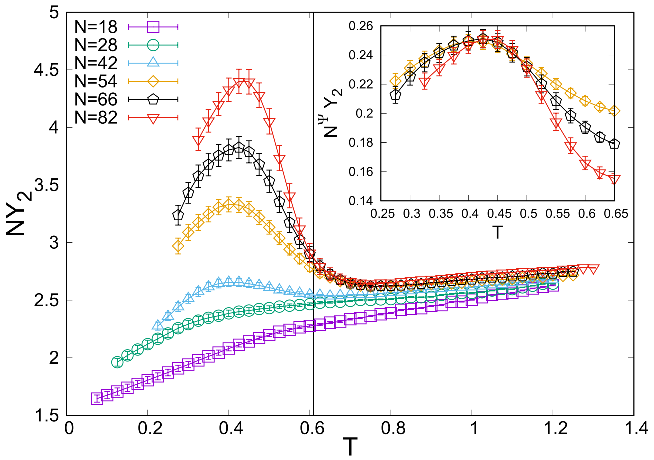
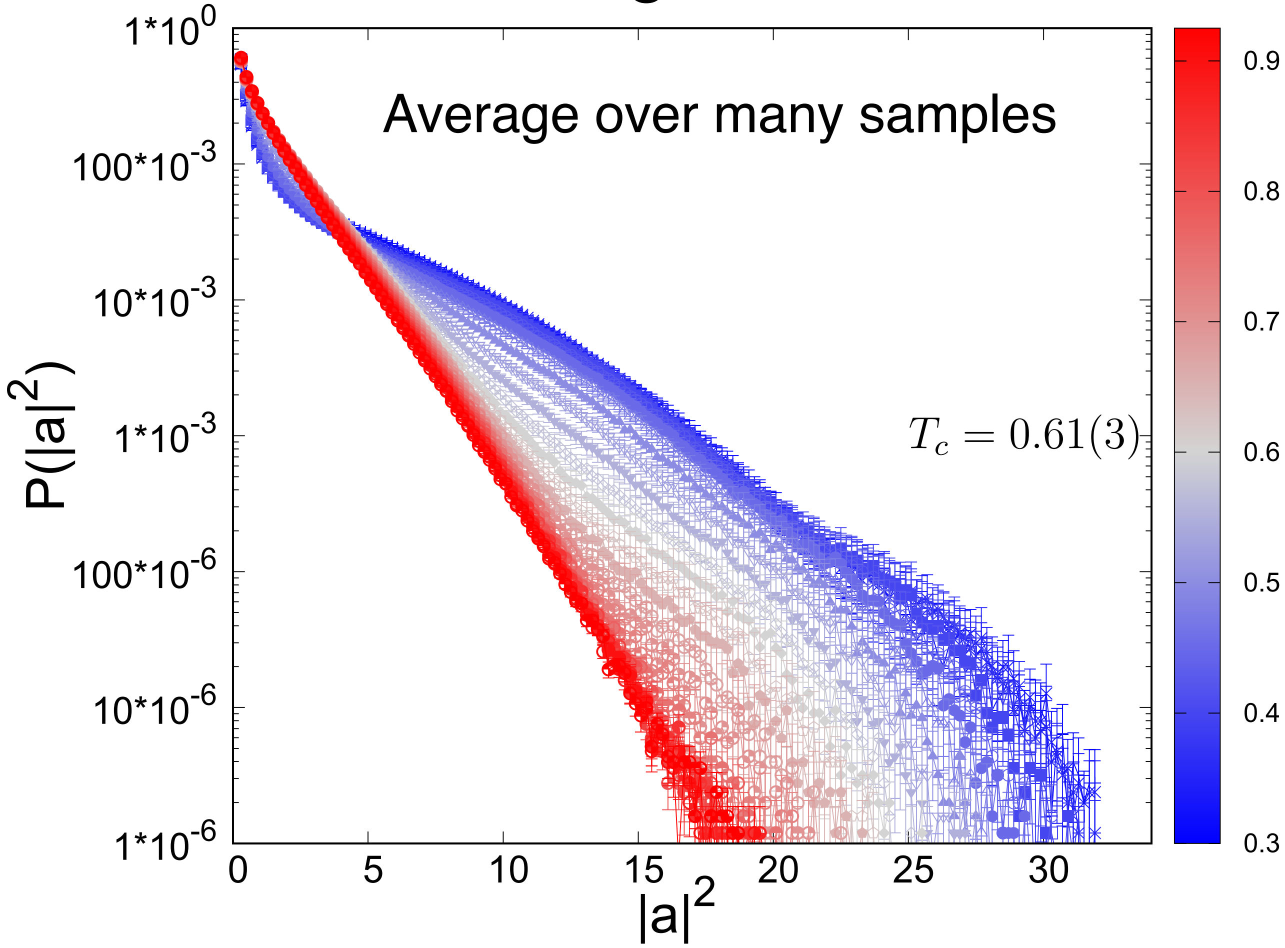
Monte Carlo simulations of the spherical 4-phasor ML RL

Intensity distribution in the pseudo-condensation regime





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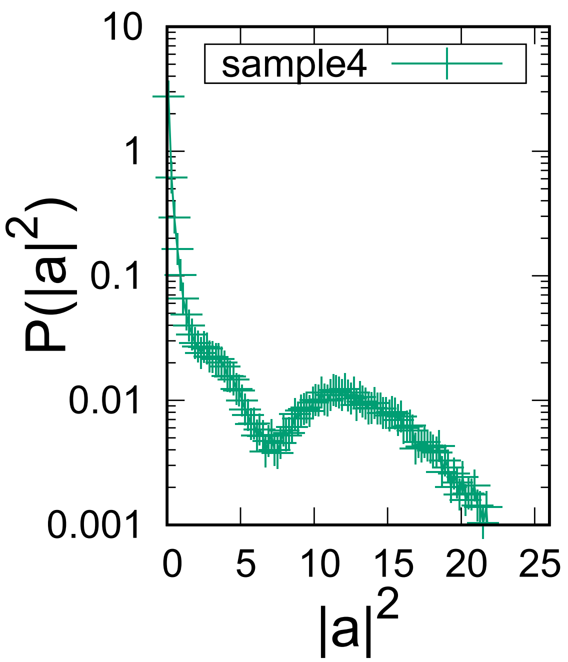
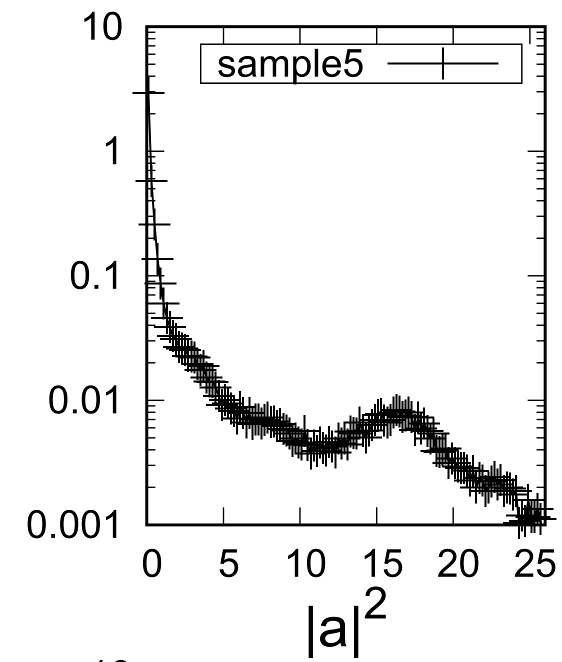
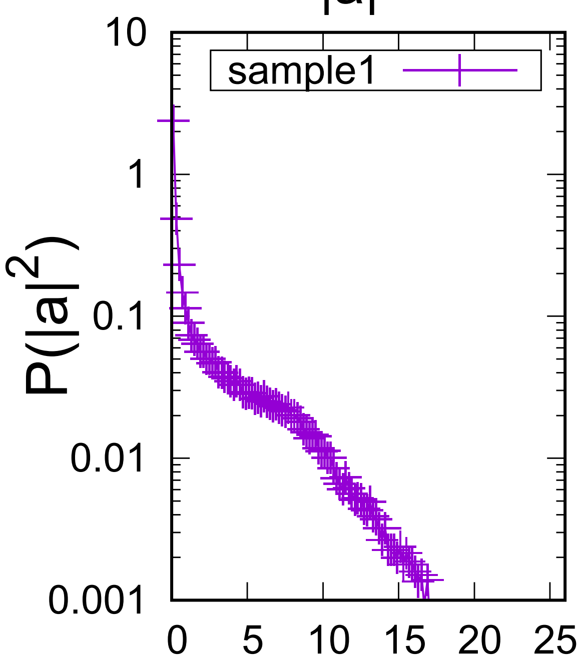
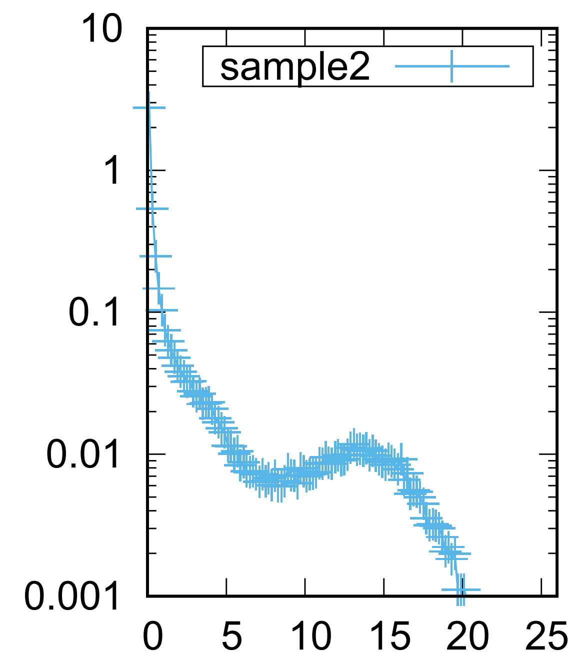
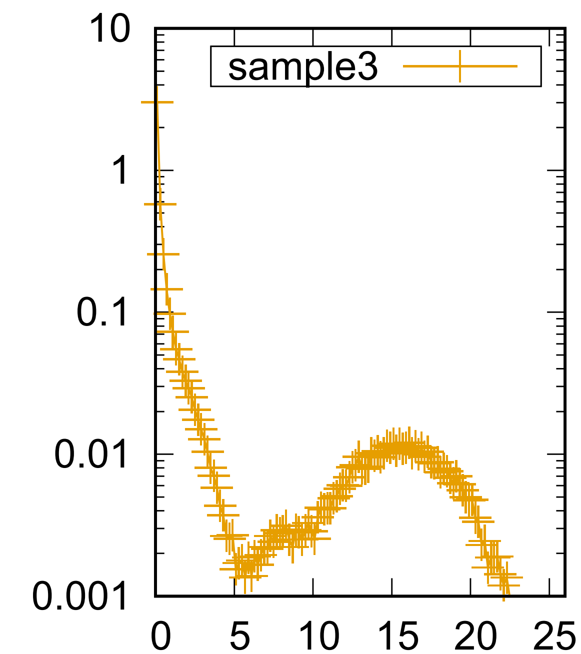




Single samples at T=0.3 (lowest)







Monte Carlo simulations of the spherical 4-phasor ML RL

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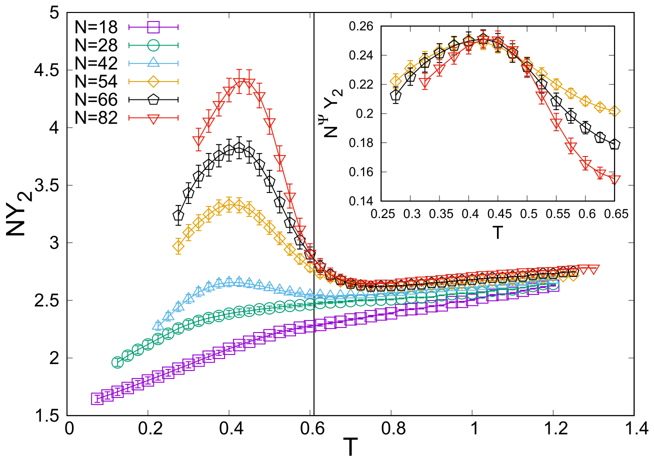

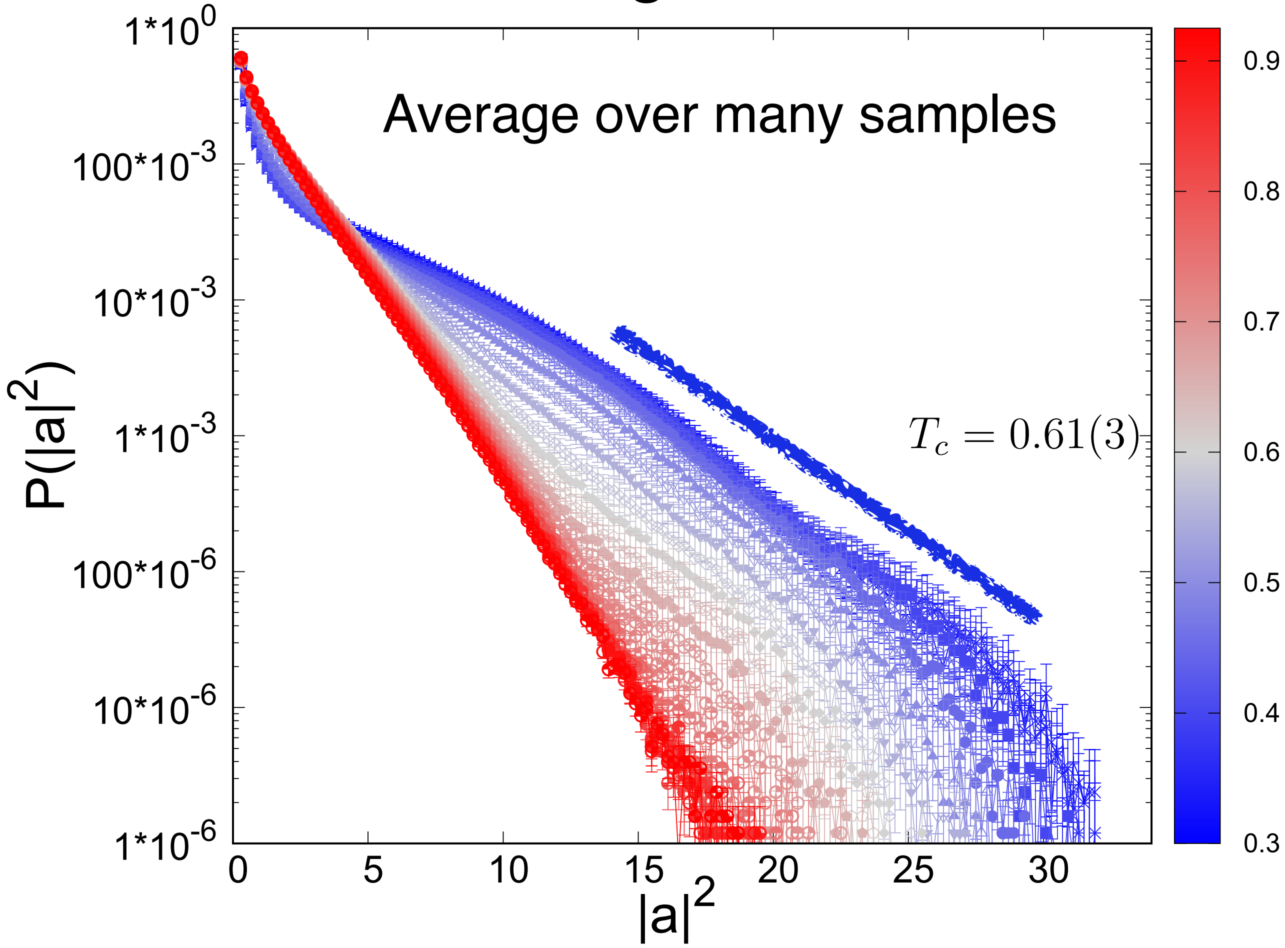

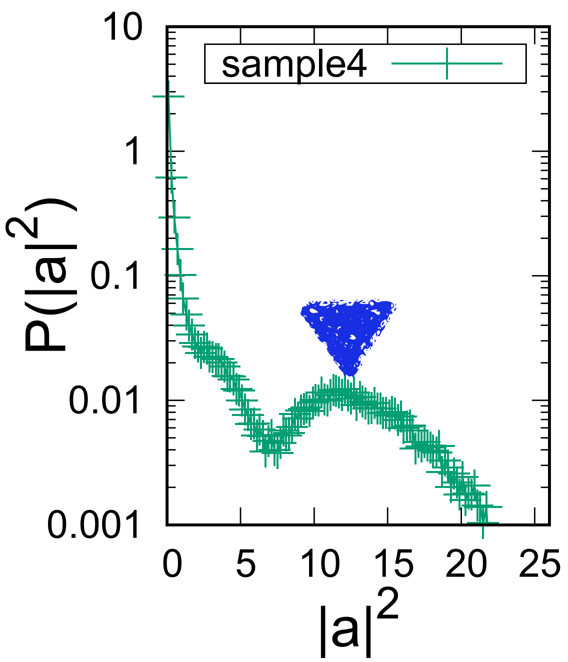

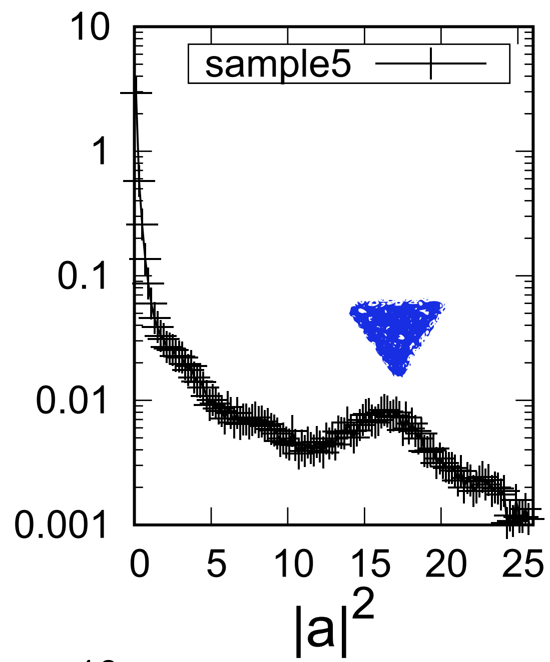

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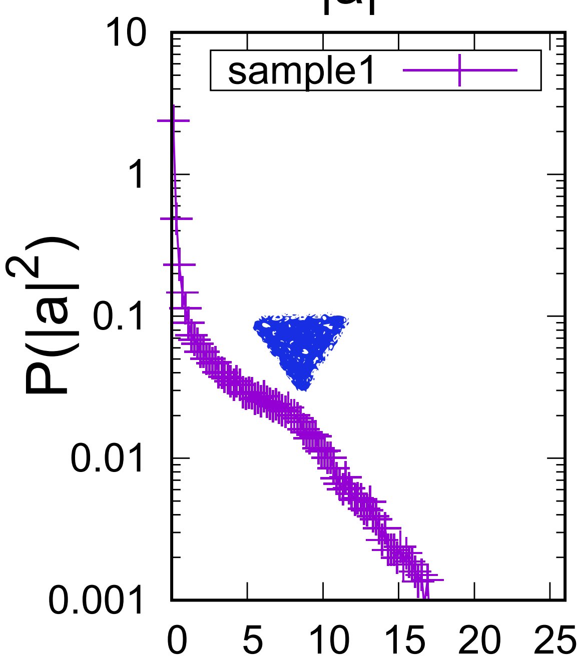
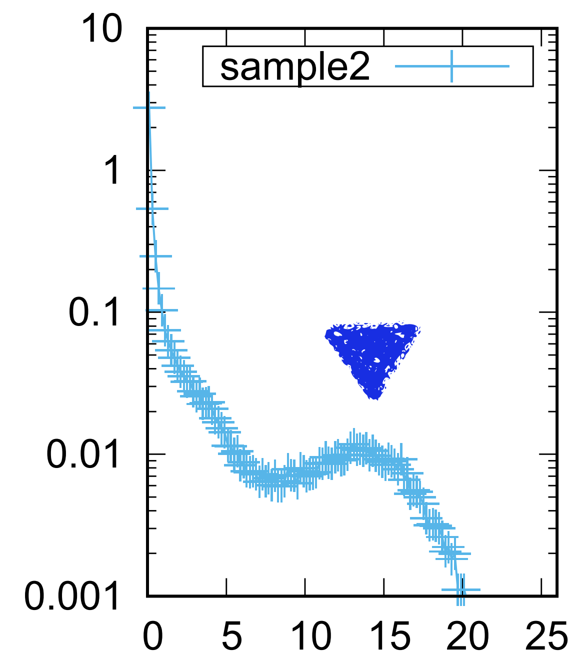
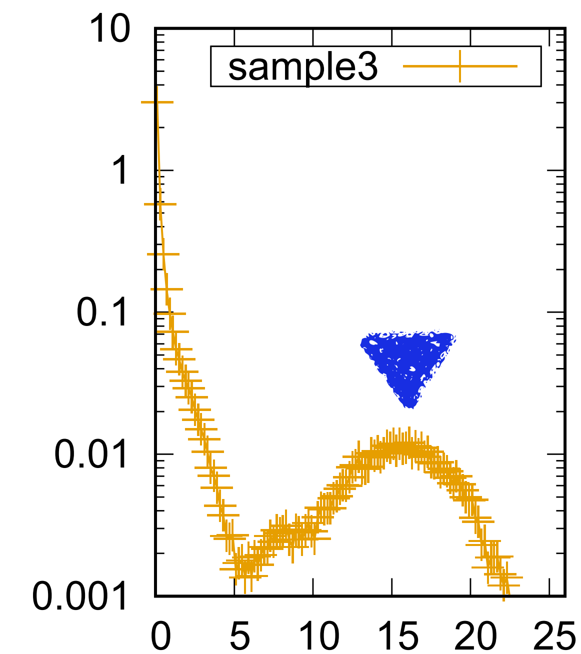






Why pseudo-condensation ?

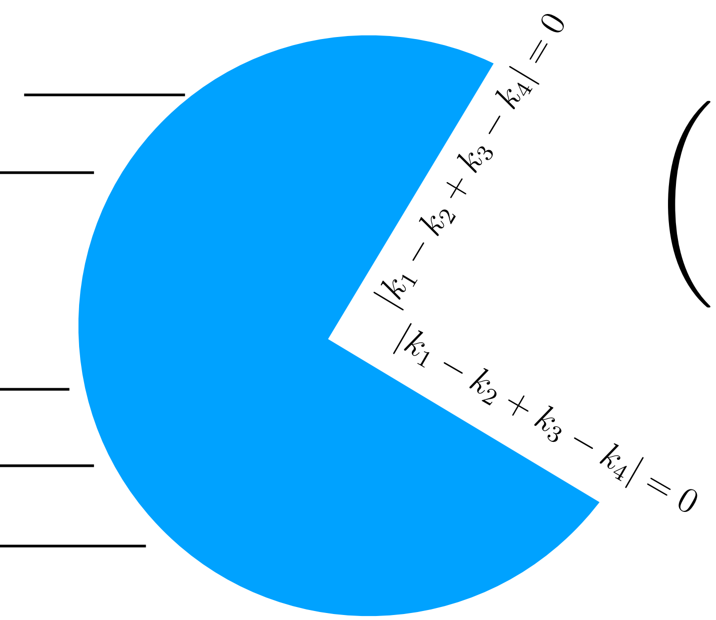
There is a connection to connectivity

$$\mathcal{H}[\boldsymbol{a}] = - \sum_{\boldsymbol{k}|\mathrm{FMC}(\boldsymbol{k})} J_{k_1 k_2 k_3 k_4} \overline{a}_{k_1} a_{k_2} \overline{a}_{k_3} a_{k_4} + \mathrm{c.c.}$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

$$\sigma_4^2 = \overline{J^2_{k_1 k_2 k_3 k_4}} \propto \frac{1}{N^2}$$

Applying the Frequency Matching Condition the interaction graph is diluted by a factor 1/N



$$\left(\begin{array}{c} N \\ 4 \end{array} \right) = \mathcal{O}\left(N^4\right)$$

A Marruzzo, P Tyagi, F Antenucci, A Pagnani, LL, Sci. Rep. 7, 3463 (2017)

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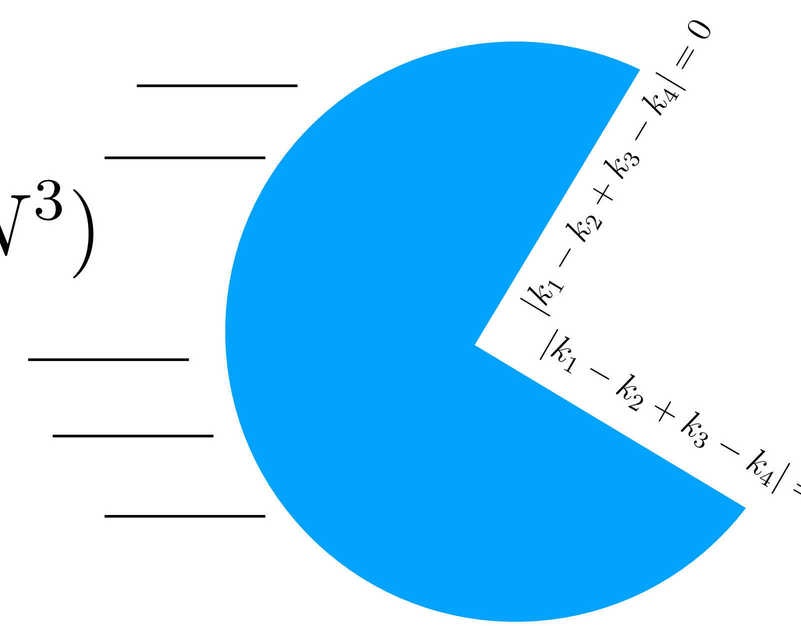
Applying the Frequency Matching Condition the interaction graph is diluted by a factor 1/N

The scaling of the number of elements in the Hamiltonian is

$$\underbrace{\begin{pmatrix} N \\ 4 \end{pmatrix}}_{\text{Complete factor graph}} \times \underbrace{\left(\frac{2}{3N} + \frac{1}{3N^3} \right)}_{\text{Frequency matching condition pruning}} = \mathcal{O}(N^3)$$



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$$\sigma_4^2 = \overline{J^2_{k_1k_2k_3k_4}} \propto \frac{1}{N^2}$$

EXTENSIVE ENERGY $\quad \min\limits_{\substack{a \\ \text{UNIFORM}}} \mathcal{H}[\boldsymbol{a}] = \mathcal{O}(N)$

This holds if $\underset{\text{UNIFORM}}{\{\boldsymbol{a}\}}$: $\quad |a_k| \simeq 1 \;\forall\; k = 1, \ldots, N$

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

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Why pseudo-condensation ?

There is a connection to connectivity

$$\mathcal{H}[\boldsymbol{a}] = -\sum_{\boldsymbol{k}|\mathrm{FMC}(\boldsymbol{k})} J_{k_1k_2k_3k_4}\overline{a}_{k_1}a_{k_2}\overline{a}_{k_3}a_{k_4} + \mathrm{c.c.}$$

$$\mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

$$\sigma_4^2 = \overline{J^2_{k_1k_2k_3k_4}} \propto \frac{1}{N^2}$$

EXTENSIVE ENERGY $\quad \min_{\substack{a \\ \text{UNIFORM}}} \mathcal{H}[\boldsymbol{a}] = \mathcal{O}(N)$

This holds if $\underset{\text{UNIFORM}}{\{\boldsymbol{a}\}}: \quad |a_k| \simeq 1 \;\forall\; k = 1, \ldots, N$

But what if condensation occurs
(on a single quadruplet)?

$$\underset{\text{CONDENSED}}{\{\boldsymbol{a}\}}: \quad |a_k| \in \square \propto \sqrt{N} \quad , \quad |a_k|_{\not\in \square} = 0$$

$$\underset{\text{CONDENSED}}{\mathcal{H}[\boldsymbol{a}]} = -J_{1234}\overline{a}_1 a_2 \overline{a}_3 a_4 = \frac{1}{\mathcal{O}(N)}\mathcal{O}(N^2) = \mathcal{O}(N)$$

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

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There is no advantage in condensation, nor in equipartition. They coexists: pseudo-condensation

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

Pseudo-condensation is a special case

The behaviour of the 4-phasor mode-locking model is borderline

$$\mathcal{H}[\boldsymbol{a}] = -\sum_{\boldsymbol{k}|\mathrm{FMC}(\boldsymbol{k})} J_{k_1k_2k_3k_4}\overline{a}_{k_1}a_{k_2}\overline{a}_{k_3}a_{k_4} + \mathrm{c.c.} \qquad \mathcal{E} = \epsilon N = \sum_{n=1}^{N} |a_n|^2$$

In the narrow band case

$$\sigma_4^2 = \overline{J^2_{k_1k_2k_3k_4}} \propto \frac{1}{N^3}$$

$$\mathcal{H}[\boldsymbol{a}] = -J_{1234}\overline{a}_1a_2\overline{a}_3a_4 = \frac{1}{\mathcal{O}(N^{3/2})}\mathcal{O}(N^2) = \mathcal{O}(N^{1/2}) \ll \mathcal{O}(N)$$

Starting with a dilute O(N^3) bipartite ER graph and pruning with FMC

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CONDENSED

Equipartition holds

Starting with a dilute O(N^3) bipartite ER graph and pruning with FMC

$$\sigma_4^2 = \overline{J^2_{k_1k_2k_3k_4}} \propto \frac{1}{N}$$

$$\mathcal{H}[\boldsymbol{a}] = -J_{1234}\overline{a}_1a_2\overline{a}_3a_4 = \frac{1}{\mathcal{O}(N^{1/2})}\mathcal{O}(N^2) = \mathcal{O}(N^{3/2}) \gg \mathcal{O}(N)$$

CONDENSED

Power condensation occurs

Ensemble equivalence breaks down!

Why pseudo-condensation ?

$$c = \frac{\#\ \text{couplings}}{\#\ \text{modes}}$$

INTENSITY EQUIPARTITION

$$|a_k| \simeq 1, \qquad \forall\ k$$

$c \sim N^4$

INTENSITY PSEUDO-CONDENSATION

$c \sim N^3$

FRUSTRATED

$c \sim N, N^2$

$$|a_{k\in\Box}| = \mathcal{O}(\sqrt{N}), \qquad |a_{k\notin\Box}| = 0$$

INTENSITY CONDENSATION

GLASSY RANDOM LASERS

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

Why pseudo-condensation ?

$$c = \frac{\#\ \text{couplings}}{\#\ \text{modes}}$$

INTENSITY EQUIPARTITION

$$|a_k| \simeq 1, \qquad \forall\ k$$

ORDERED OR NON-GLASSY RANDOM LASERS

UNFRUSTRATED

$c \sim N^3, N^4$

$c \sim N^2$

$c \sim N$

FRUSTRATED

$c \sim N^4$

$c \sim N^3$

$c \sim N, N^2$

GLASSY RANDOM LASERS

INTENSITY PSEUDO-CONDENSATION

$$|a_{k\in\square}| = \mathcal{O}(\sqrt{N}), \qquad |a_{k\notin\square}| = 0$$

INTENSITY CONDENSATION

F Antenucci, M Ibanez Berganza, LL,
PRB 92, 014204 (2015), PRA 91, 043811 (2015)

J. Niedda, G. Gradenigo, LL,
JSTAT 053302 (2023)

Why pseudo-condensation ?

For any p-interacting number of modes, any spherical p-spin model

$$c = \frac{\#\text{ couplings}}{\#\text{ modes}}$$

INTENSITY EQUIPARTITION

$$|a_k| \simeq 1, \qquad \forall\, k$$

UNFRUSTRATED

$c \sim N^{>p/2}$

$c \sim N^{p/2}$

$c \sim N^{<p/2}$

FRUSTRATED

$c \sim N^{p}$

$c \sim N^{<p-1}$

$c \sim N^{p-1}$

INTENSITY PSEUDO-CONDENSATION

$$|a_{k\in\square}| = \mathcal{O}(\sqrt{N}), \qquad |a_{k\notin\square}| = 0$$

INTENSITY CONDENSATION

J. Niedda, G. Gradenigo, LL, JSTAT 053302 (2023)

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Conclusions

- Random lasers can be effectively modelled by statistical mechanics of disordered systems (mode-locking, spectral properties, lasing threshold,…)
- Some random lasers display glassy (multi-equilibria) features and allow for directly measuring the RSB order parameter distribution
- Monte Carlo dynamics of the leading model, the spherical 4-phasor mode-locked model, yield evidence for

    Mean-field universality class for the lasing critical point

    RSB at high pumping/low temperature

    and pseudo-condensation of the overall intensity (connection with connectivity of the interaction network)

- In progress:

  * Monte Carlo simulations at and off-equilibrium with continuous band of frequencies, real material gain profiles, losses (M Benedetti, G Trinca-Cintioli, J Niedda)

  * Interpolation between analytic and experimental IFO, out of equilibrium effects (G Trinca Cintioli, J Niedda)

  * Analytic theory for spin-glasses on mode-locked graphs — merit factor revisited (J Niedda, G Parisi)

  * Mode-locked random lasers on sparse graphs avoiding condensation (M Benedetti)

  * Measurements campaign on different random laser compounds (solids, viscous liquids, organic, inorganic, 2D, 3D, varying mode extensions, ..) probed by different pumping lasers (ns, ps, fs pulses) to tune disorder, understand the relationship between mode extension and coupling network connectivity, deepen the reliability of photonics measure of the Parisi P(q) (CNR-NANOTEC, D Sanvitto, L De Marco, I Viola, M De Giorgi, …)

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Direzione Generale della Ricerca del MUR

Post-doc positions opening soon on the MUR-funded project
Statics and Dynamics of Spin Glass and Disordered Systems:
Theory and Numerics to Understand Experiments.

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Random lasers as complex disordered systems

Luca Leuzzi

Lucabib

J Niedda, G Gradenigo, LL, G Parisi, SciPost Physics 14, 144 (2023)
J Niedda, LL, G Gradenigo JSTAT 053302 (2023)
C Conti, N Ghofraniha, LL, G Ruocco, in “Spin Glass Theory and Far Beyond: Replica Symmetry Breaking After 40 Years”, pp. 307-334 (World Scientific, 2023), arXiv:2209.03781
F Antenucci, G Lerario, B Silva Fernandéz, L De Marco, M De Giorgi, D Ballarini, D Sanvitto, LL, PRL 126, 173901 (2021)
G Gradenigo, F Antenucci, LL PRR 2, 023399 (2020)
A Marruzzo, P Tyagi, F Antenucci, A Pagnani, LL, SciPost Phys. 5, 002 (2018)
I Viola, LL, C Conti, N Ghofraniha, in “Organic Lasers”, pp. 151-192 (Jenny Stanford, 2018)
A Marruzzo, P Tyagi, F Antenucci, A Pagnani, LL, Sci. Rep. 7, 3463 (2017)
F Antenucci, “Statistical Physics of Wave Interactions: A Unified Approach to Mode-Locking and Random Lasers” (Springer, 2016)
P Tyagi, A Marruzzo, A Pagnani, F Antenucci, LL, PRB 94, 024203 (2016)
A Marruzzo, LL, PRB 93, 094206 (2016)
F Antenucci, A Crisanti, M Ibáñez-Berganza, A Marruzzo, LL, Phil. Mag. 96, 704-731 (2016)
F Antenucci, A Crisanti, LL, Sci. Rep. 5, 16792 (2015)
F Antenucci, M Ibanez Berganza, LL, PRB 92, 014204 (2015)
P Tyagi, A Pagnani, F Antenucci, M Ibanez Berganza, LL, JSTAT P05031 (2015)
F Antenucci, A Crisanti, LL, PRA 91, 053816 (2015)
F Antenucci, M Ibanez Berganza, LL, PRA 91, 043811 (2015)
A Marruzzo, LL, PRB 91, 054201 (2015)
N Ghofraniha, I Viola, F Di Maria, G Barbarella, G Gigli, LL, C Conti, Nat. Commun. 6, 6058 (2015)
F Antenucci, C Conti, A Crisanti, LL, PRL 114, 043901 (2015)
V Folli, N Ghofraniha, A Puglisi, LL, C Conti, Sci. Rep. 3, 2251 (2013)
V Folli, A Puglisi, LL, C Conti, PRL 108, 248002 (2012)
C Conti, LL, PRB B 83, 134204 (2011)
LL, C Conti, V Folli, L Angelani, G Ruocco, PRL 102, 083901 (2009)

FUNDING AGENCIES

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SEVENTH FRAMEWORK PROGRAMME

MARIE CURIE ACTIONS

FUTURO IN RICERCA

CNRNANOTEC INSTITUTE OF NANOTECHNOLOGY

Eigenmode basis in open cavity

$$\boldsymbol{E}(\boldsymbol{r},t) \;=\; \sum_n a_n(t)\boldsymbol{E}_n(\boldsymbol{r})e^{-\imath\omega_n t} + \text{ c.c.}$$

OPEN CAVITY:

(i) Mirror cavities with leakages: there will be also radiative modes, whose frequencies take values in a continuous dominion. Different modes can have the same frequency.

(ii) Mirror-less lasers in random media, with inhomogeneous optical susceptibility profiles. Discrete lasing frequencies will not be all equispaced and may overlap. Furthermore, the "optical cycle" and the "roundtrip time" are not defined. Their random analogues depend on the scatterers structure.

Non-diagonal linear contribution to the Hamiltonian.

What is a complete basis in an open system?

Fox-Li modes, quasi-bound states, *constant flux modes*, ….

Strong Interactions in Multimode Random Lasers
Türeci, Ge, Rotter, Douglas Stone 2008

Feschbach projection onto radiative and localized mode subspaces
Hackenbroich, Viviescas, Haken 2003

Random graph ML laser

$$\mathcal{H} = -\sum_{n=1}^{N} g_n^{(0)} |a_n|^2 - J \sum_{\omega_{n_1}-\omega_{n_2}+\omega_{n_3}-\omega_{n_4}=0} a_{n_1} a_{n_2}^* a_{n_3} a_{n_4}^* + \text{c.c.} \quad \Rightarrow \quad \mathcal{H} = -\sum_{jklm}^{\text{ML}} J_{jklm} \cos(\phi_j - \phi_k + \phi_l - \phi_m)$$

quenched amplitude or amplitude equipartition $|a|$~1 : 4-XY phase model

A. Marruzzo and LL, PRB 2015

ML: $|\omega_j - \omega_k + \omega_l - \omega_m| \leqslant \gamma$

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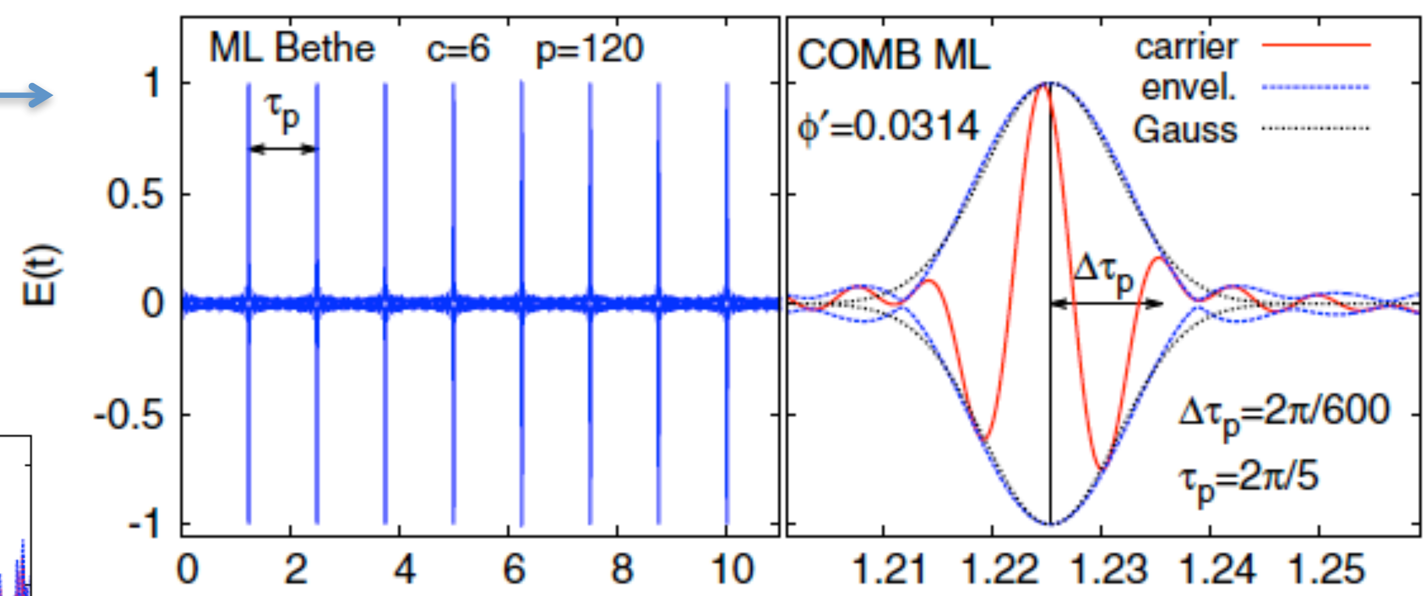
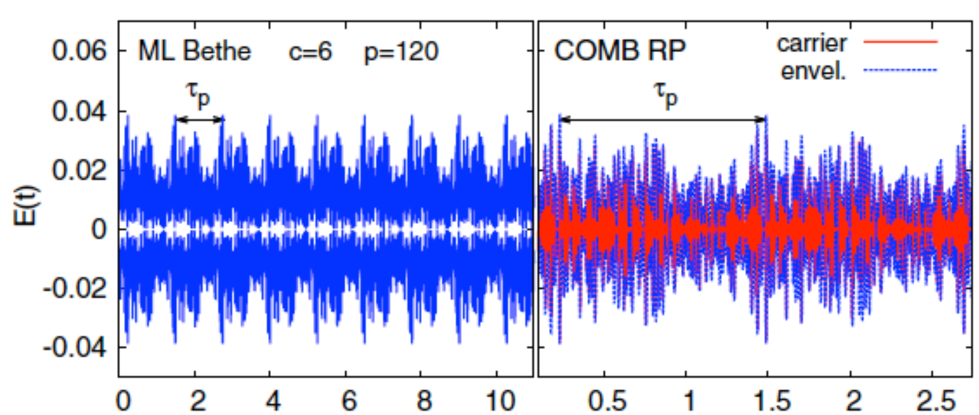
High power mode-locking

$$\phi(\omega) \simeq \phi(\omega_0) + \phi' \times (\omega - \omega_0) \longrightarrow$$





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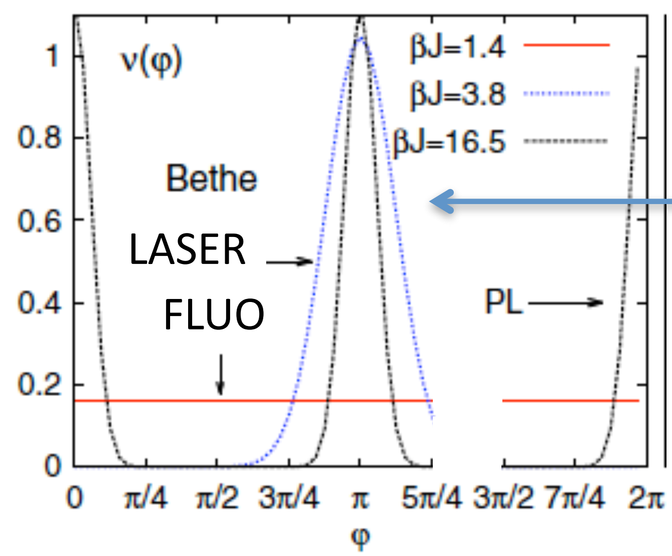
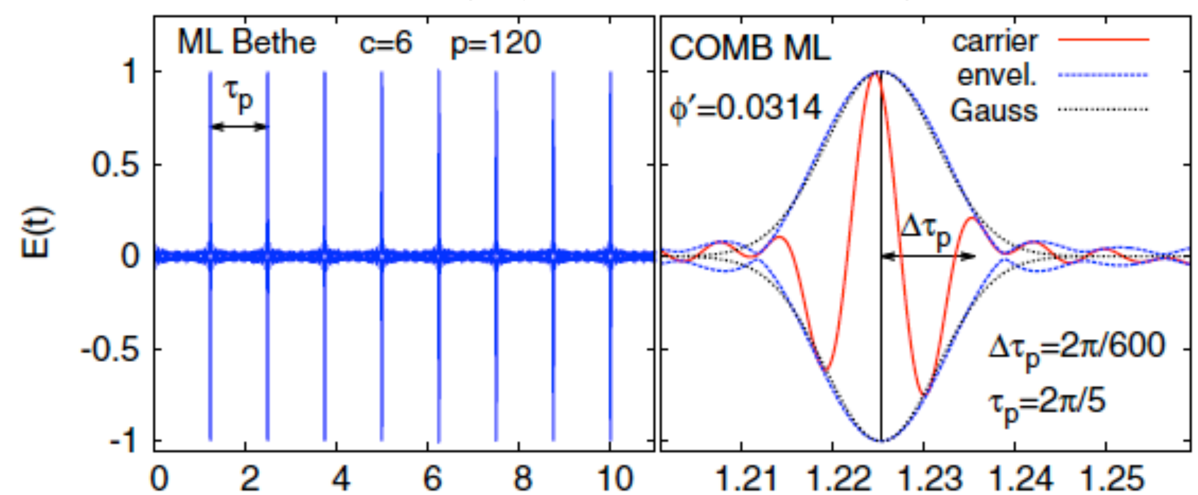
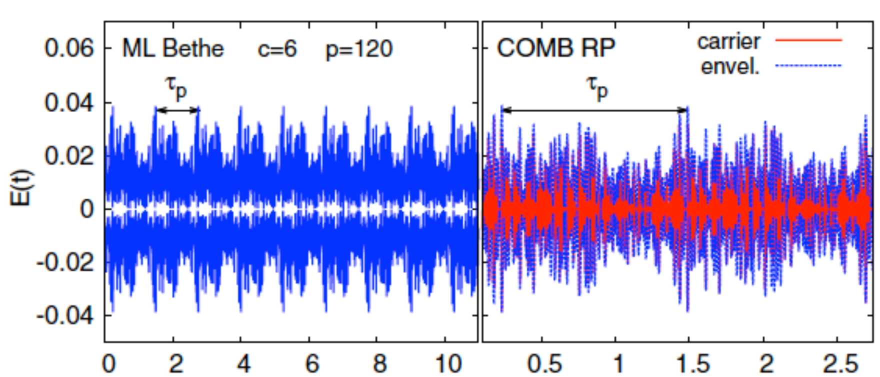


The optical regime is characterized by the phase *marginal*: the distribution of the phase values ν(ϕ)

$$\phi(\omega) \simeq \phi(\omega_0) + \phi' \times (\omega - \omega_0)$$

High power mode-locking





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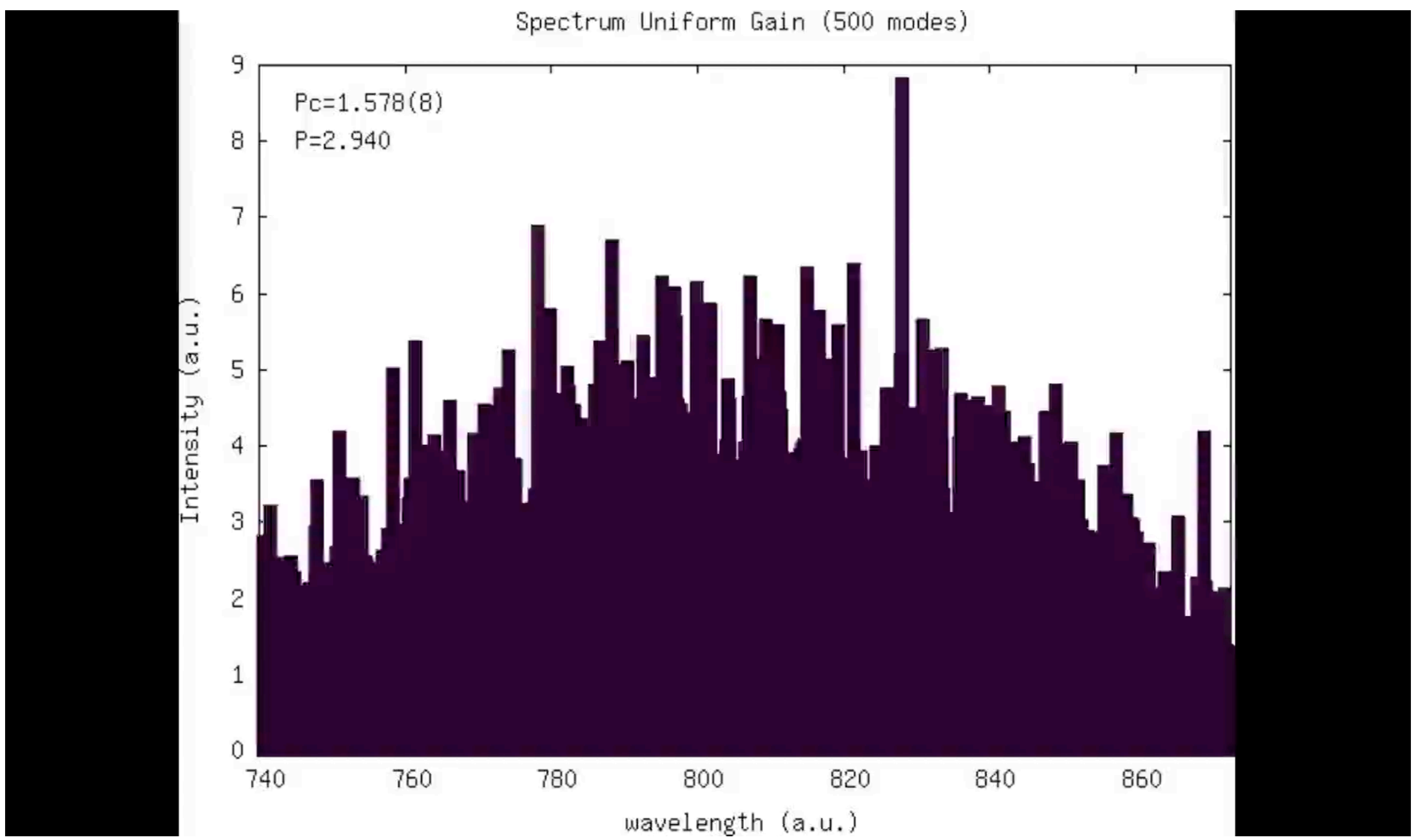
Loopy random graph ML lasers

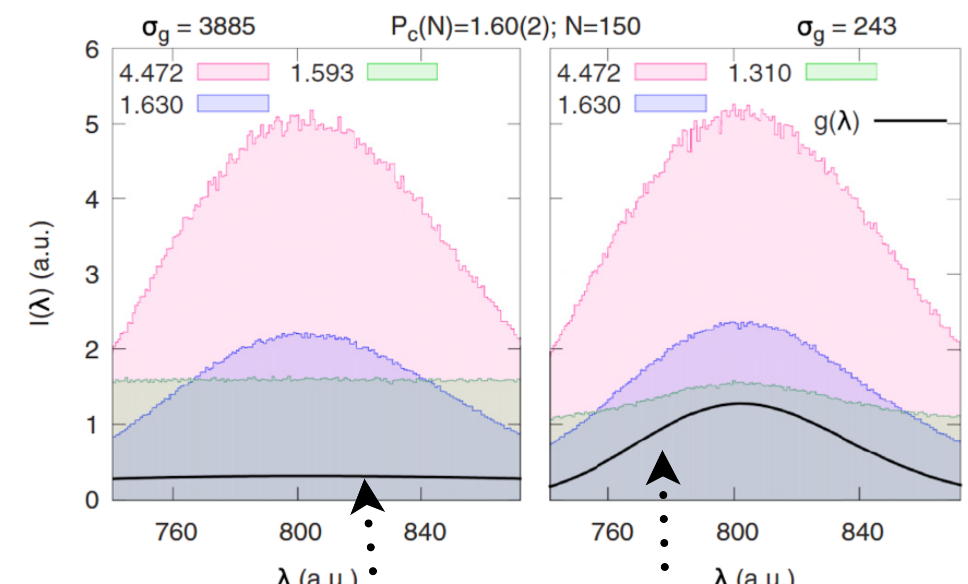
Beyond mean-field approximations we study mode-locking systems with non-trivial gain profile and interaction networks: Monte Carlo simulations.

$$\mathcal{H} = -\sum_{n=1}^{N} g_n^{(0)} |a_n|^2 - J \sum_{\omega_{n_1} - \omega_{n_2} + \omega_{n_3} - \omega_{n_4} = 0} a_{n_1} a_{n_2}^* a_{n_3} a_{n_4}^* + \text{c.c.}$$

Single pulse spectra



Integrated spectra



small signal gain $g_n^{(0)}$ spectral profile

F. Antenucci, M. Ibañez Berganza, LL, PRA 2015, PRB 2015

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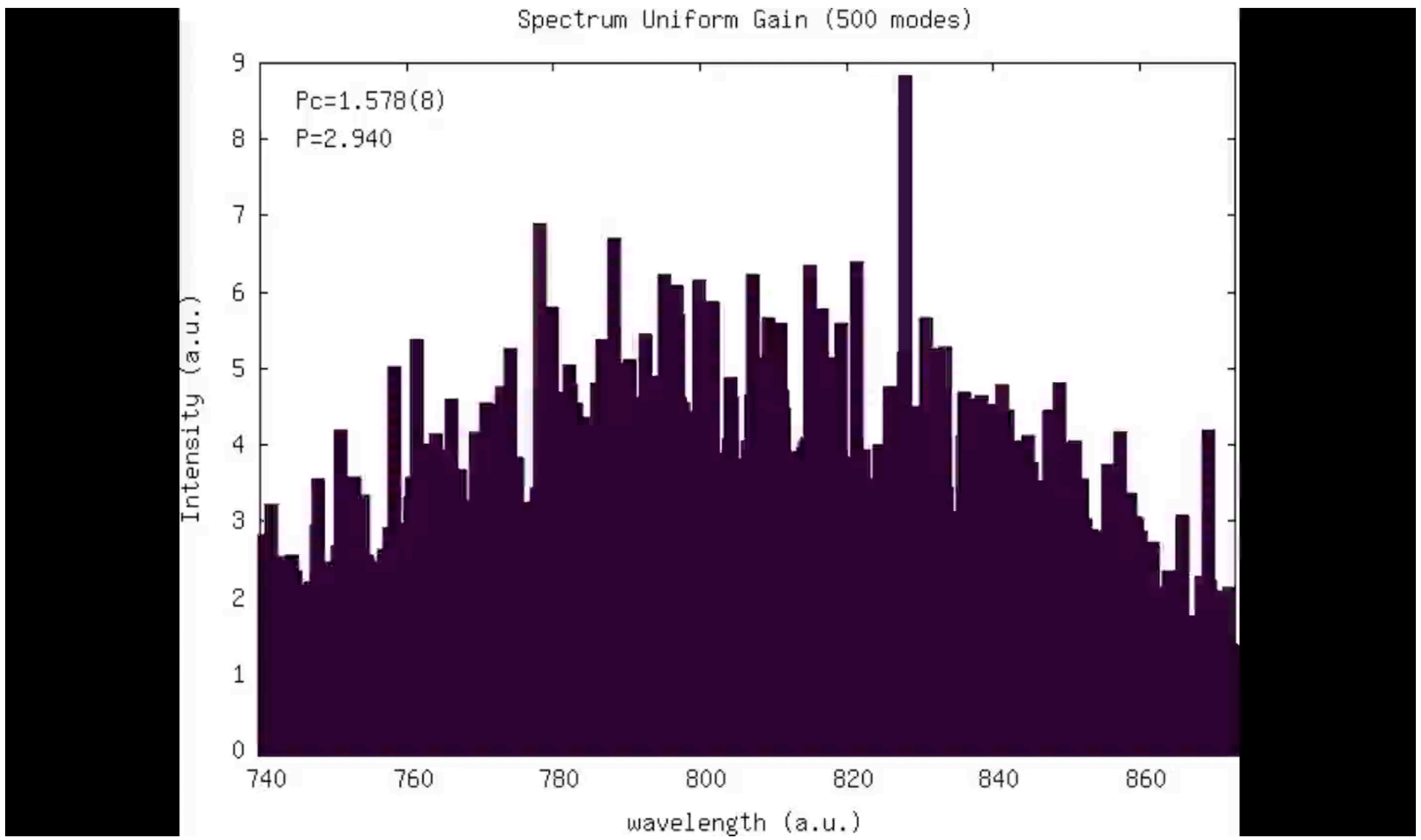
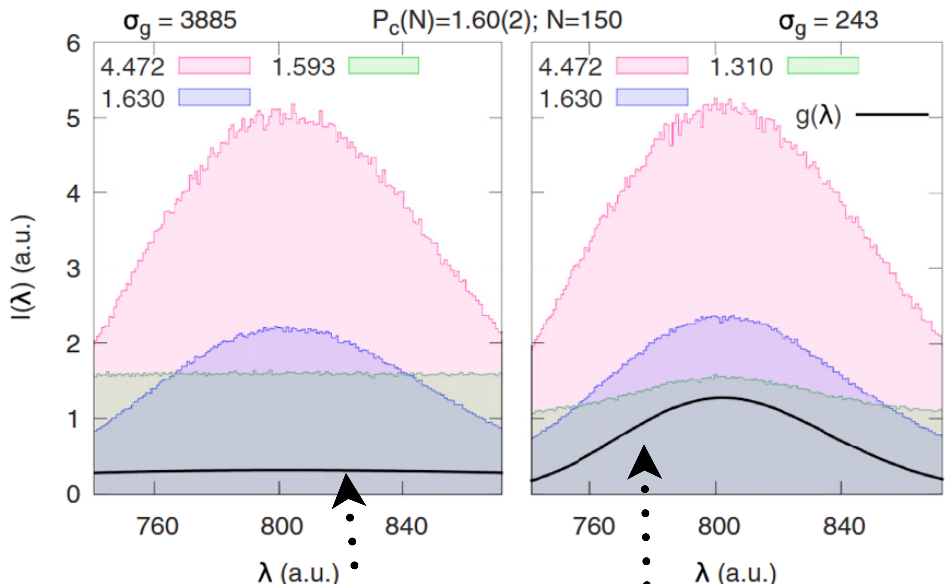


Integrated spectra



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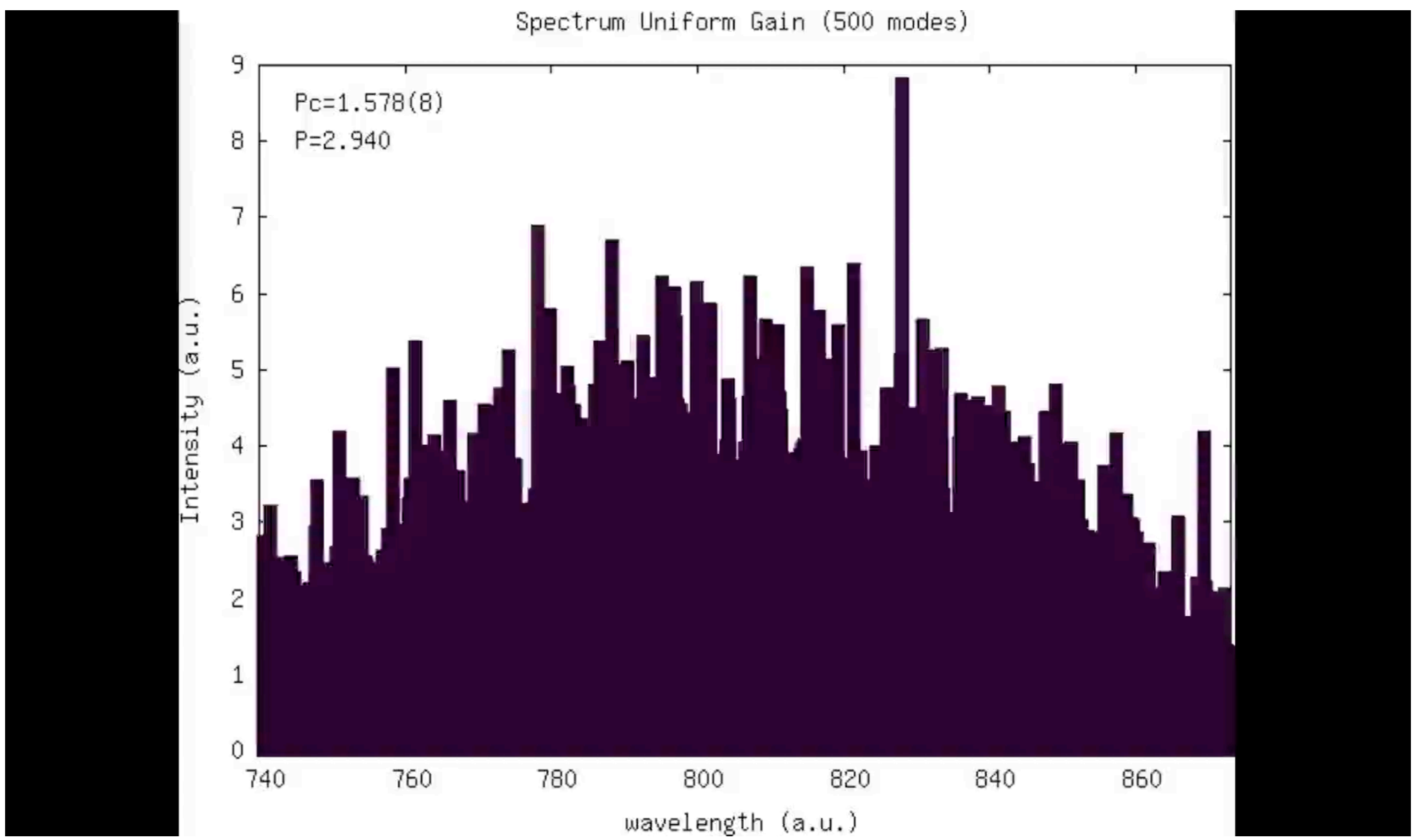

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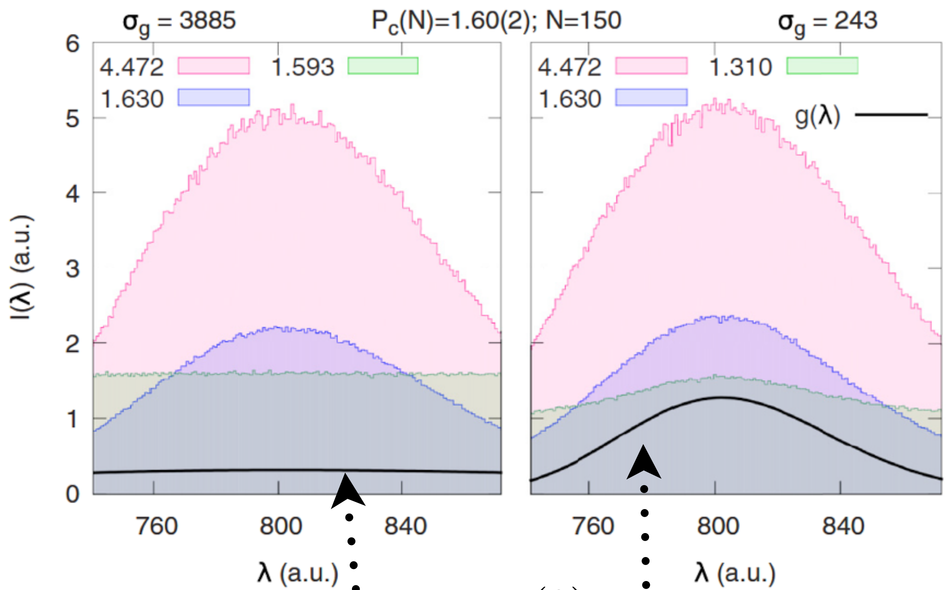

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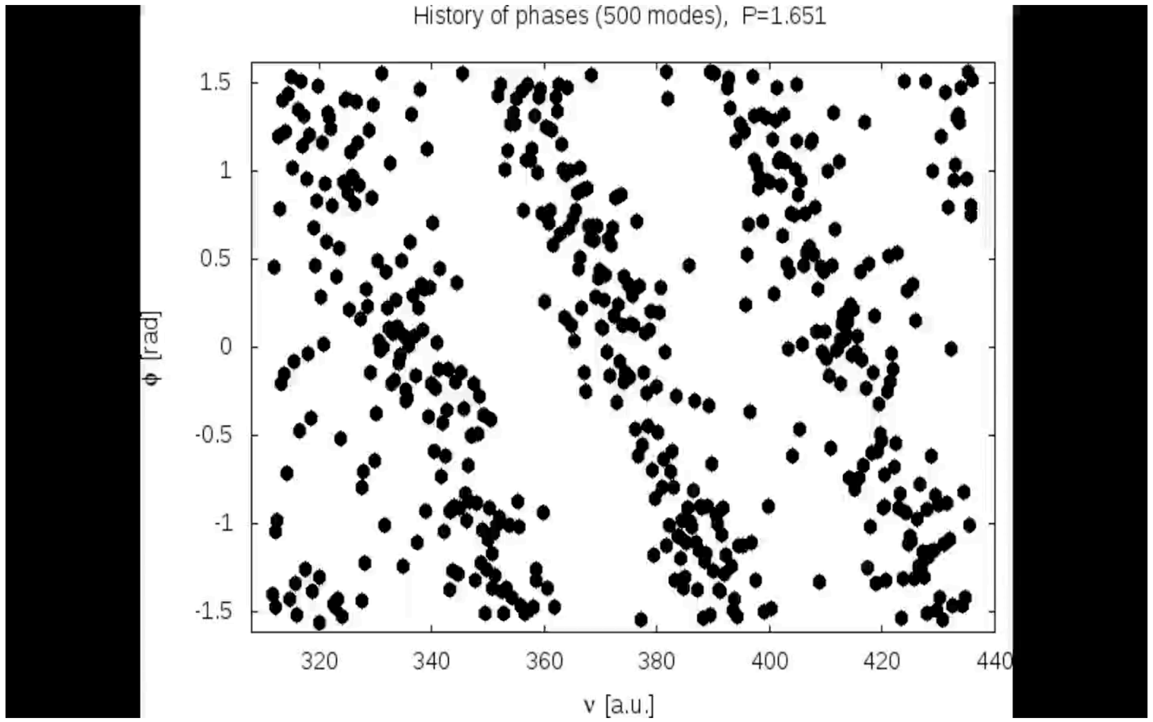

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F. Antenucci, M. Ibañez Berganza, LL, PRA 2015, PRB 2015

Mode locking

$$\phi(\omega) \simeq \phi(\omega_0) + \phi' \times (\omega - \omega_0)$$

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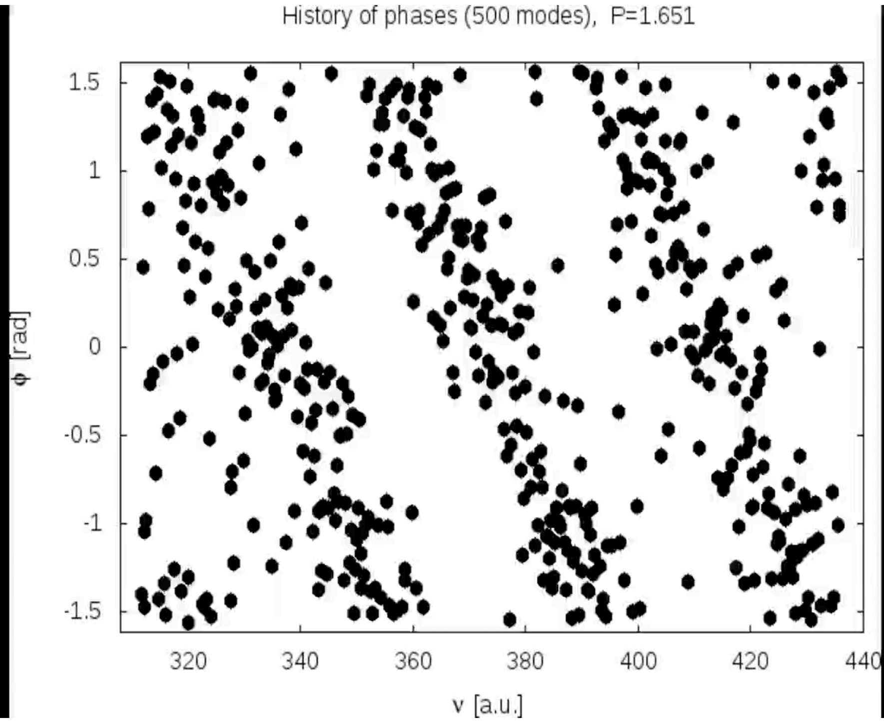

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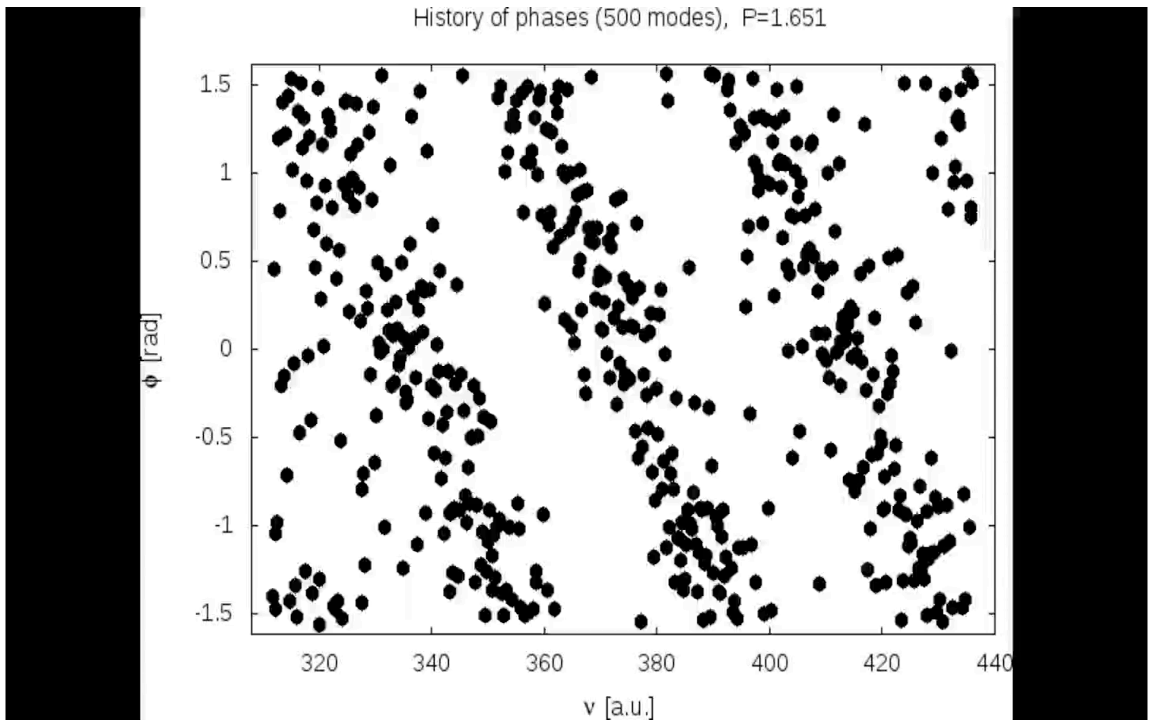

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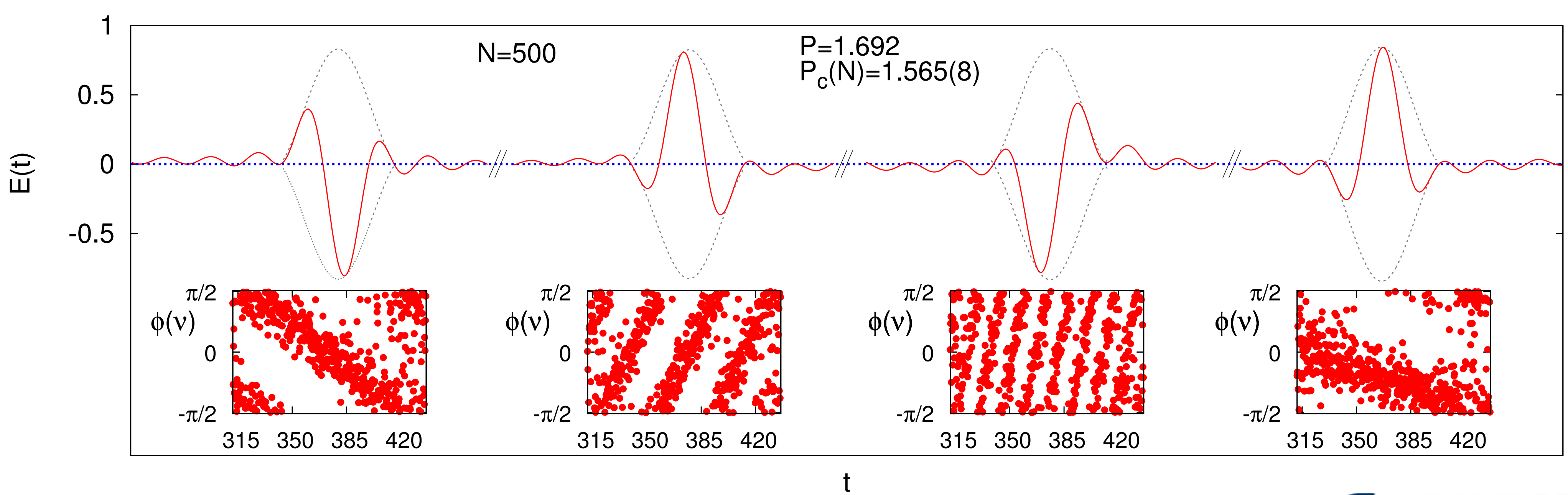

Demonstration of Self-Starting Nonlinear Mode Locking in Random Lasers

Fabrizio Antenucci,[1,4] Giovanni Lerario,[2] Blanca Silva Fernandéz,[2] Luisa De Marco,[2] Milena De Giorgi,[2]
Dario Ballarini,[2] Daniele Sanvitto,[2,*] and Luca Leuzzi[1,3,†]
[1] *CNR-NANOTEC, Institute of Nanotechnology, Soft and Living Matter Laboratory, Piazzale Aldo Moro 5, I-00185 Rome, Italy*
[2] *CNR-NANOTEC, Institute of Nanotechnology, Via Monteroni, I-73100 Lecce, Italy*
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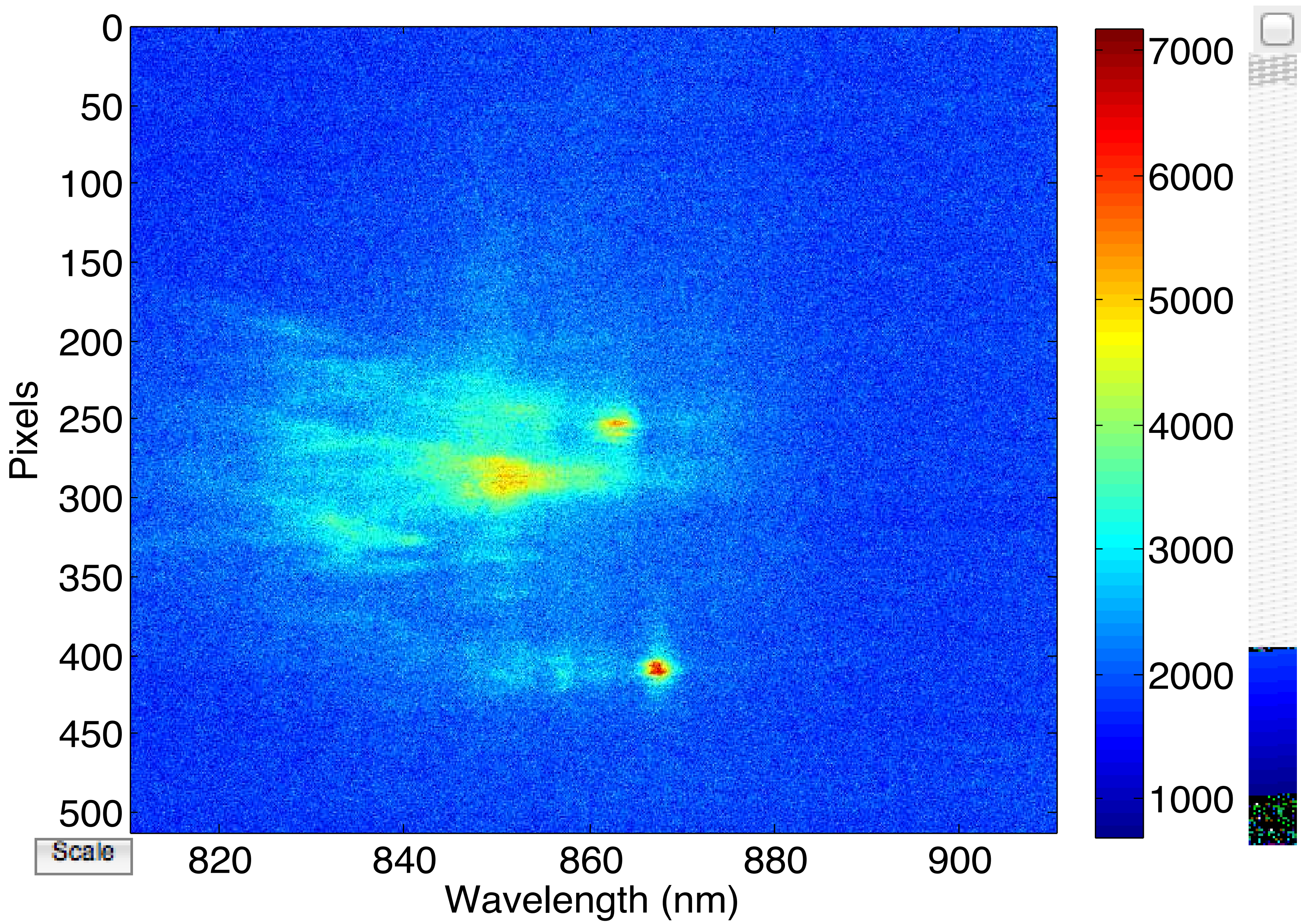
[4] *Saddle Point Science Ltd, 71 OAKS Avenue, Worcester Park KT4 8XE, United Kingdom*

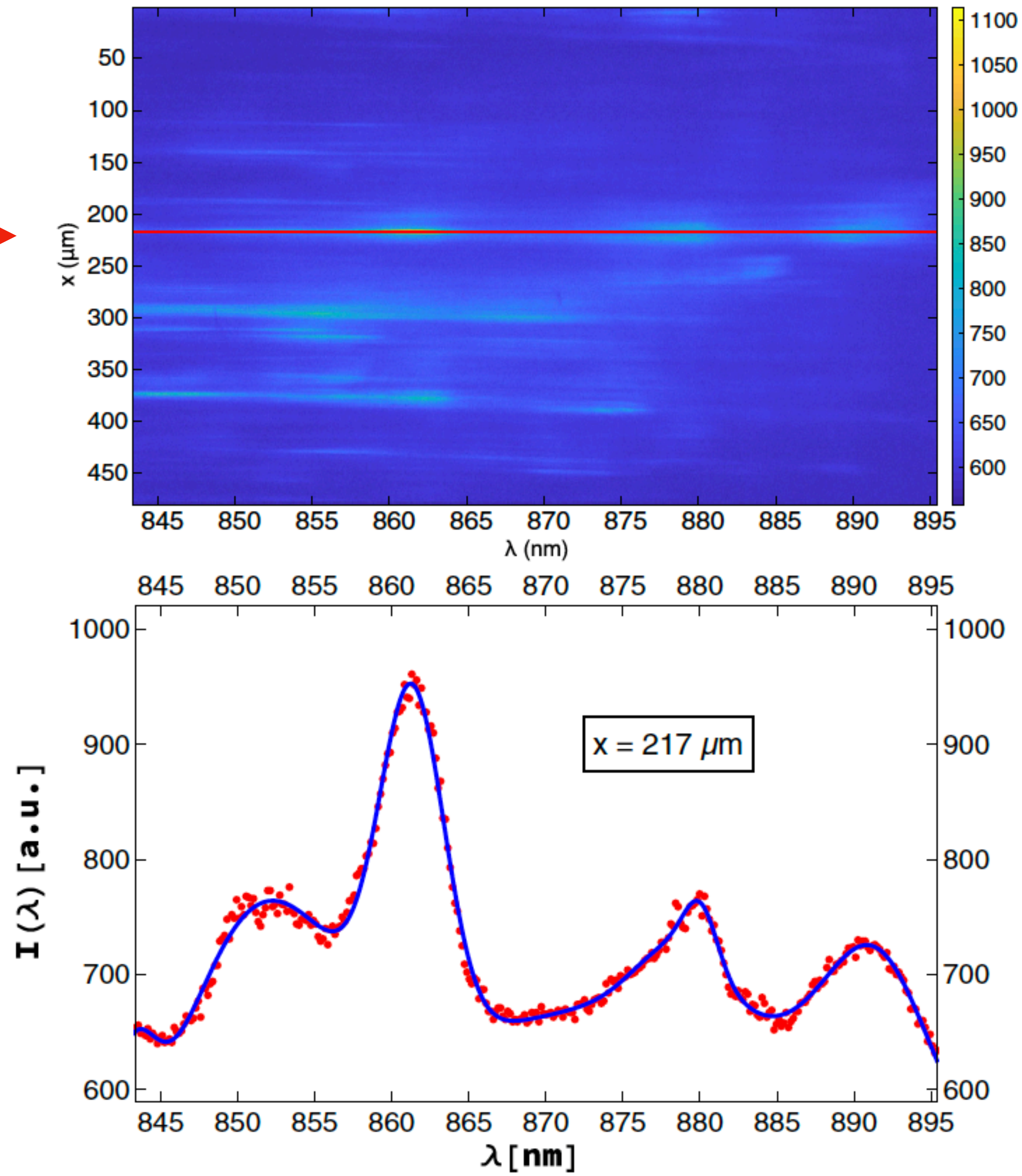
The key point is to have contemporarily the spectrum and the position of the resonance



F Antenucci et al., PRL 126, 173901 (2021)

Self-starting mode-locking in random lasers

Modes identification



Multiple Gaussian interpolation of thousands of spectra
+
Akaike inference citerion to limit overfitting

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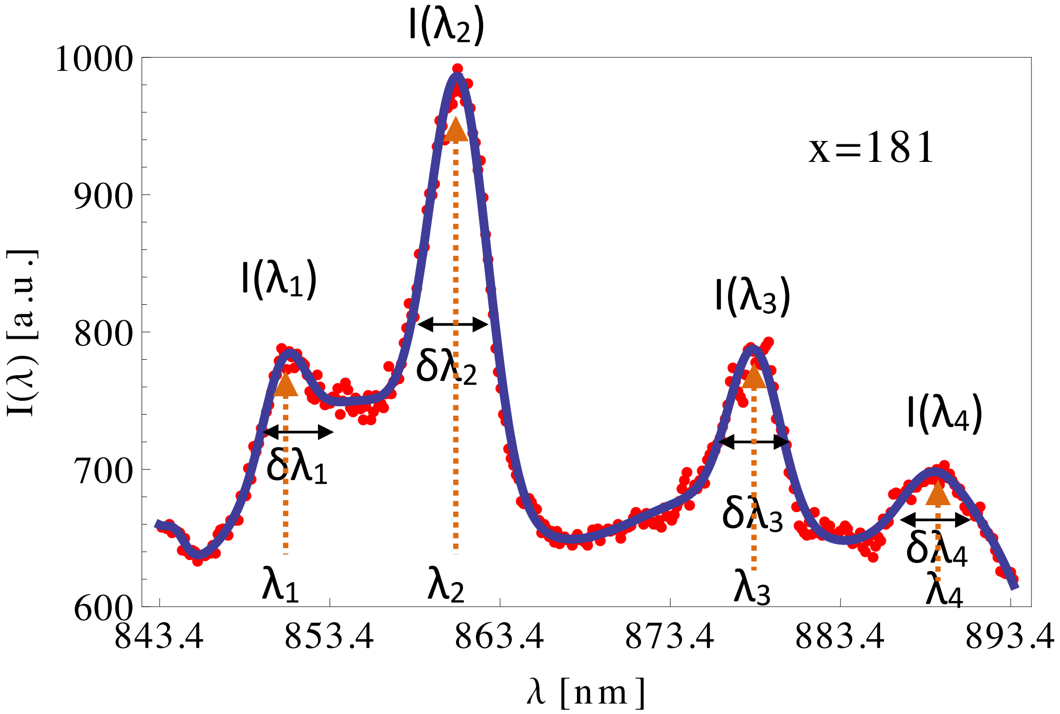
Four peaks, intensities, wavelengths, line-widths



Self-starting mode-locking in random lasers

Modes 4-point correlation

$$C_4(\omega_j,\omega_k,\omega_l,\omega_m) = \langle I_j I_k I_l I_m \rangle_c = C_4^{(0)} - C_4^{(1)} + 2C_4^{(2)} - 6C_4^{(3)}$$

$$\begin{aligned}
C_4^{(0)} &= \langle I_j I_k I_l I_m \rangle \\
C_4^{(1)} &= \langle I_j I_k I_l \rangle \langle I_m \rangle + \langle I_j I_k I_m \rangle \langle I_l \rangle + \langle I_j I_m I_l \rangle \langle I_k \rangle + \langle I_m I_k I_l \rangle \langle I_j \rangle \\
&\quad + \langle I_j I_k \rangle \langle I_l I_m \rangle + \langle I_j I_l \rangle \langle I_k I_m \rangle + \langle I_j I_m \rangle \langle I_k I_l \rangle \\
C_4^{(2)} &= \langle I_j I_k \rangle \langle I_l \rangle \langle I_m \rangle + \langle I_j I_l \rangle \langle I_k \rangle \langle I_m \rangle + \langle I_j I_m \rangle \langle I_k \rangle \langle I_l \rangle \\
&\quad + \langle I_k I_l \rangle \langle I_j \rangle \langle I_m \rangle + \langle I_k I_m \rangle \langle I_j \rangle \langle I_l \rangle + \langle I_l I_m \rangle \langle I_j \rangle \langle I_k \rangle \\
C_4^{(3)} &= -6 \langle I_j \rangle \langle I_k \rangle \langle I_l \rangle \langle I_m \rangle
\end{aligned}$$

$$c_4(\omega_j,\omega_k,\omega_l,\omega_m) \equiv \frac{C_4(\omega_j,\omega_k,\omega_l,\omega_m)}{\sigma_j(\omega_j)\sigma_k(\omega_k)\sigma_l(\omega_l)\sigma_m(\omega_m)}$$

$$\sigma_j(\omega_j) = \sqrt{\langle (I_j - \langle I_j \rangle)^2 \rangle}$$

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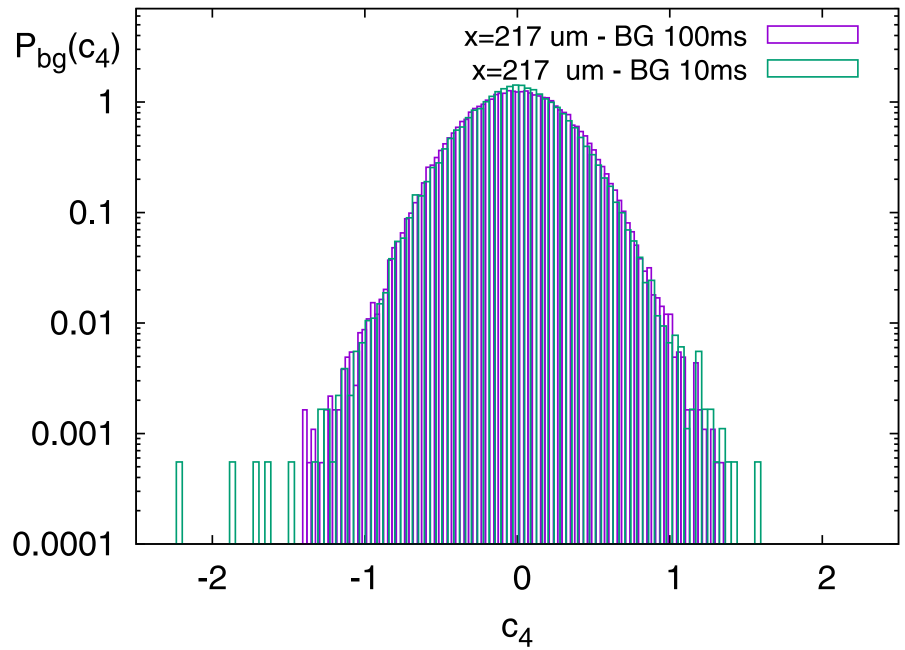
4-point correlation distributions



4-point correlations among resonances in different spectra

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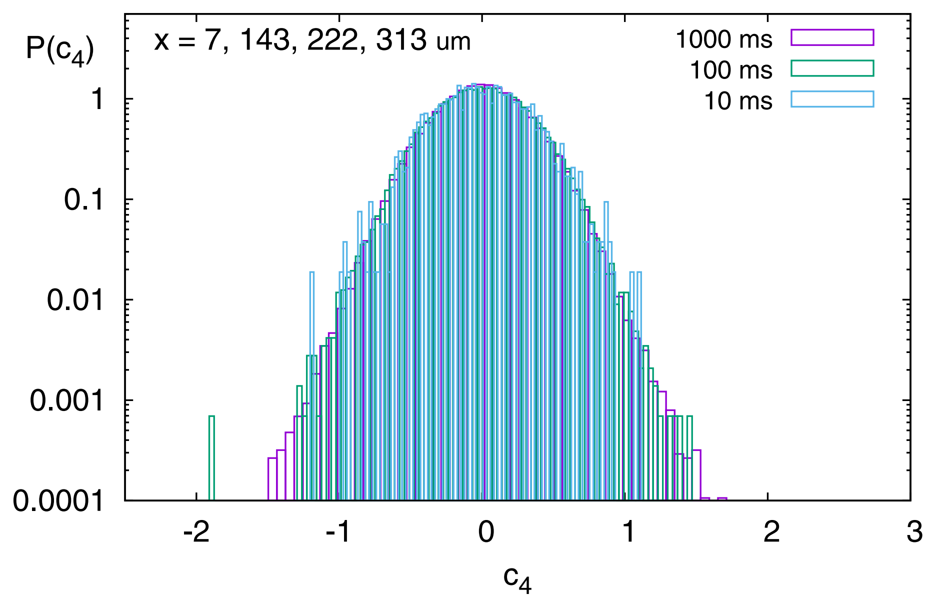
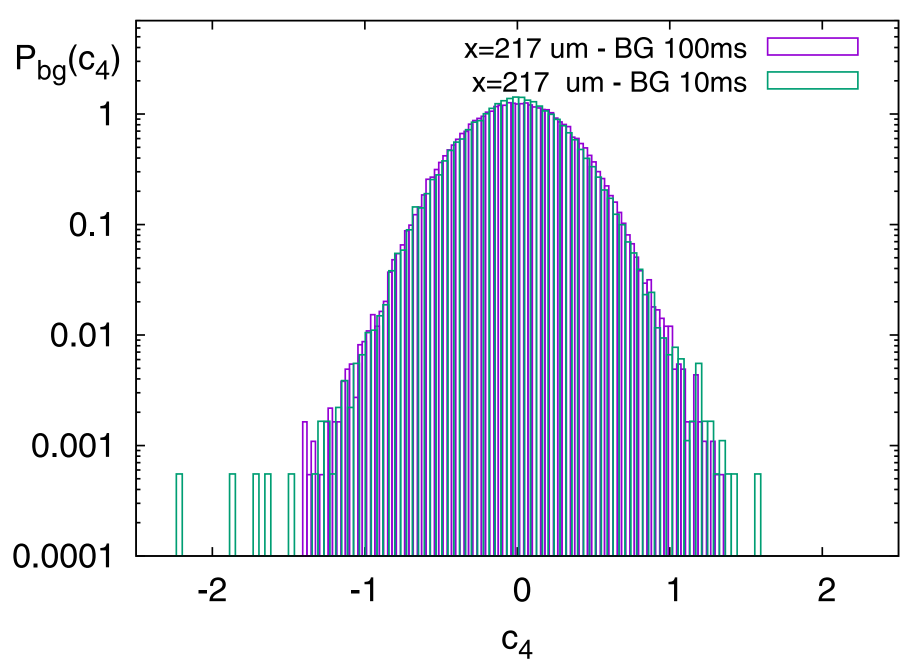
Self-starting mode-locking in random lasers

Self-starting mode-locking in random lasers



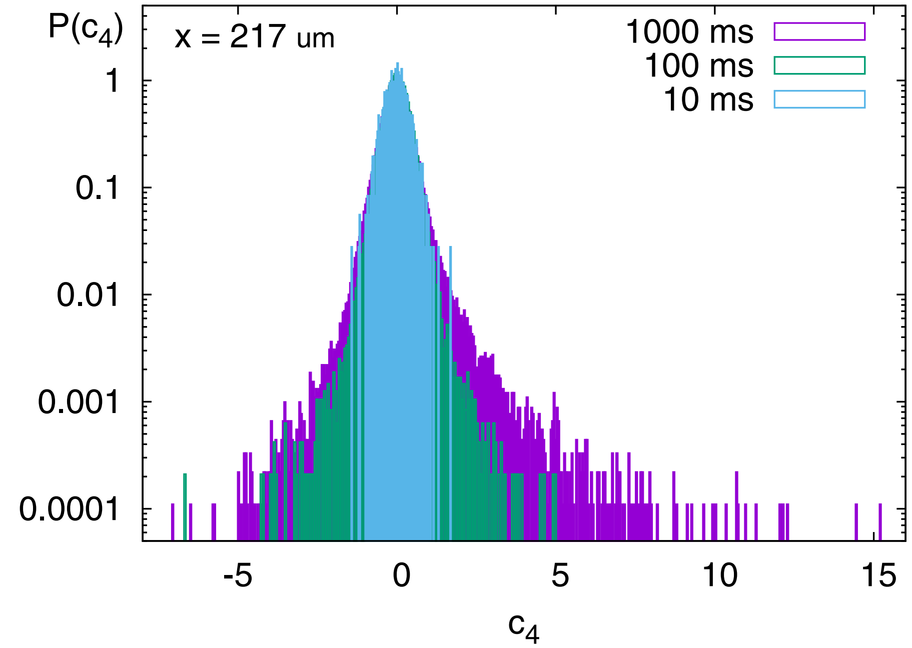
4-point correlations among resonances in the same spectrum at far away distances

4-point correlation distributions

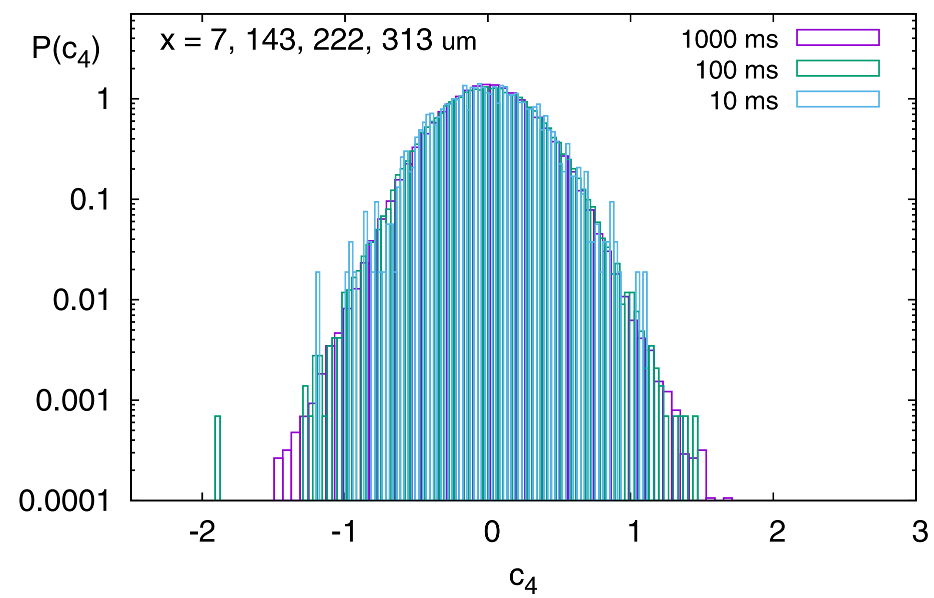


4-point correlations among resonances in different spectra

Self-starting mode-locking in random lasers

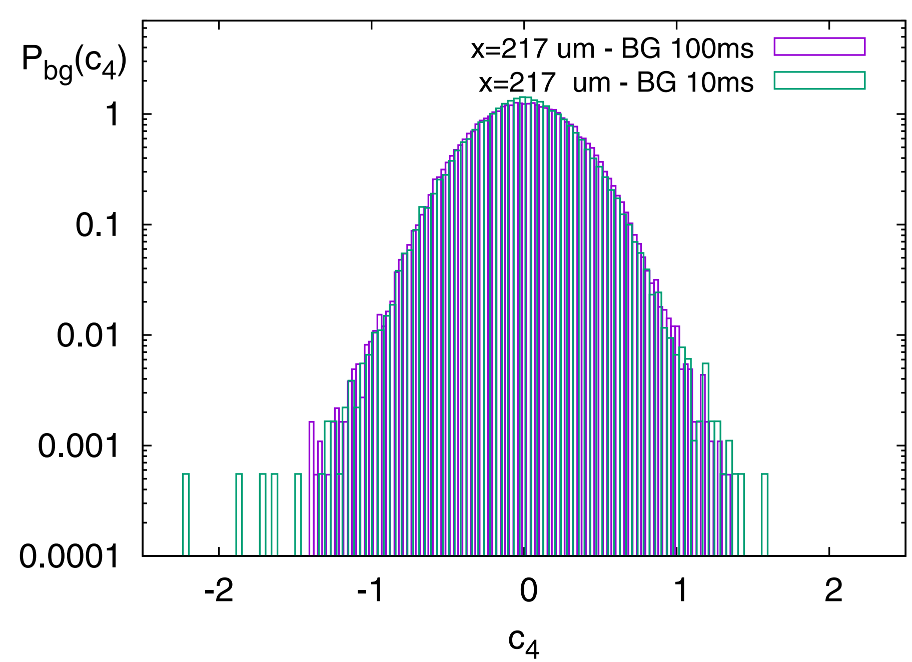


4-point correlations among resonances in the same spectrum at the same position



4-point correlations among resonances in the same spectrum at far away distances

4-point correlation distributions



4-point correlations among resonances in different spectra

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Self-starting mode-locking in random lasers

Correlation vs the mode locking condition on the matching of mode frequencies

$$|\omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_{k_4}| < \gamma; \qquad \gamma \equiv \sum_{j=1}^{4} \gamma_{k_j}$$

We introduce the control parameter

$$\Delta_4 \equiv \frac{|\omega_1 - \omega_2 + \omega_3 - \omega_4|}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}$$

The smaller, the more locked are the modes

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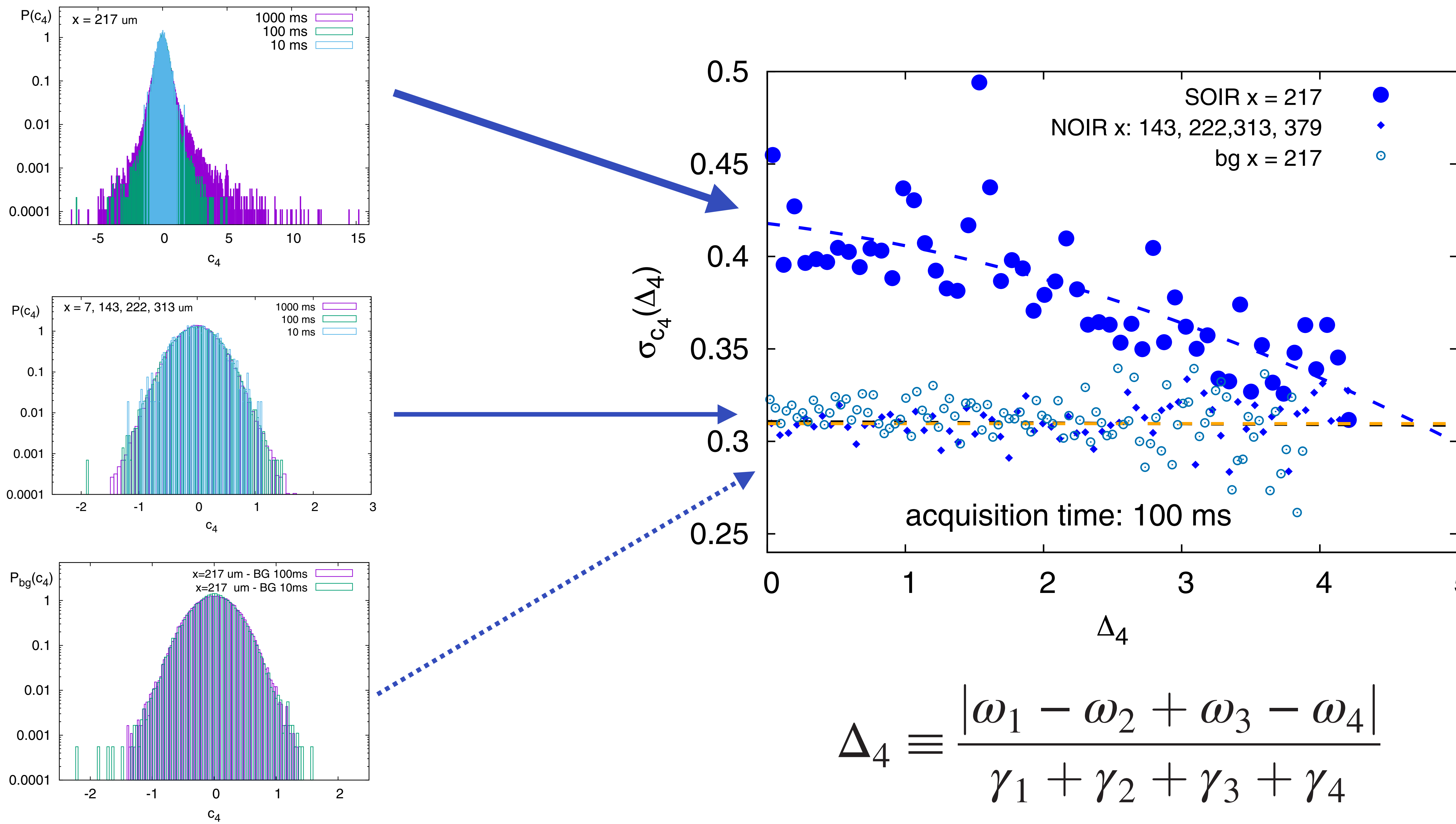
Variance correlation vs the matching of mode frequencies



$$\Delta_4 \equiv \frac{|\omega_1 - \omega_2 + \omega_3 - \omega_4|}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}$$

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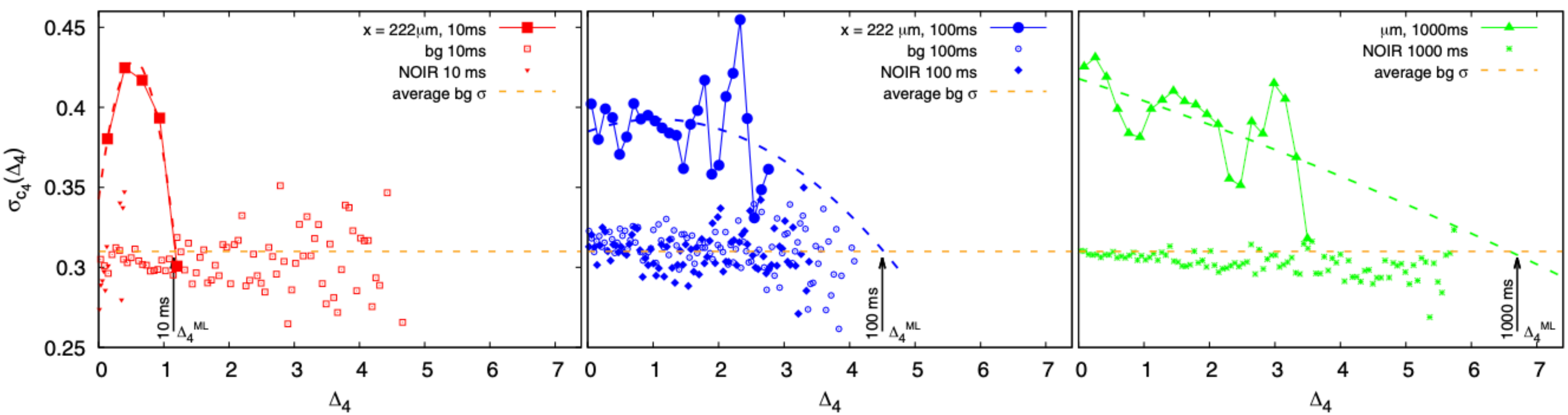
Variance correlation vs the matching of mode frequencies



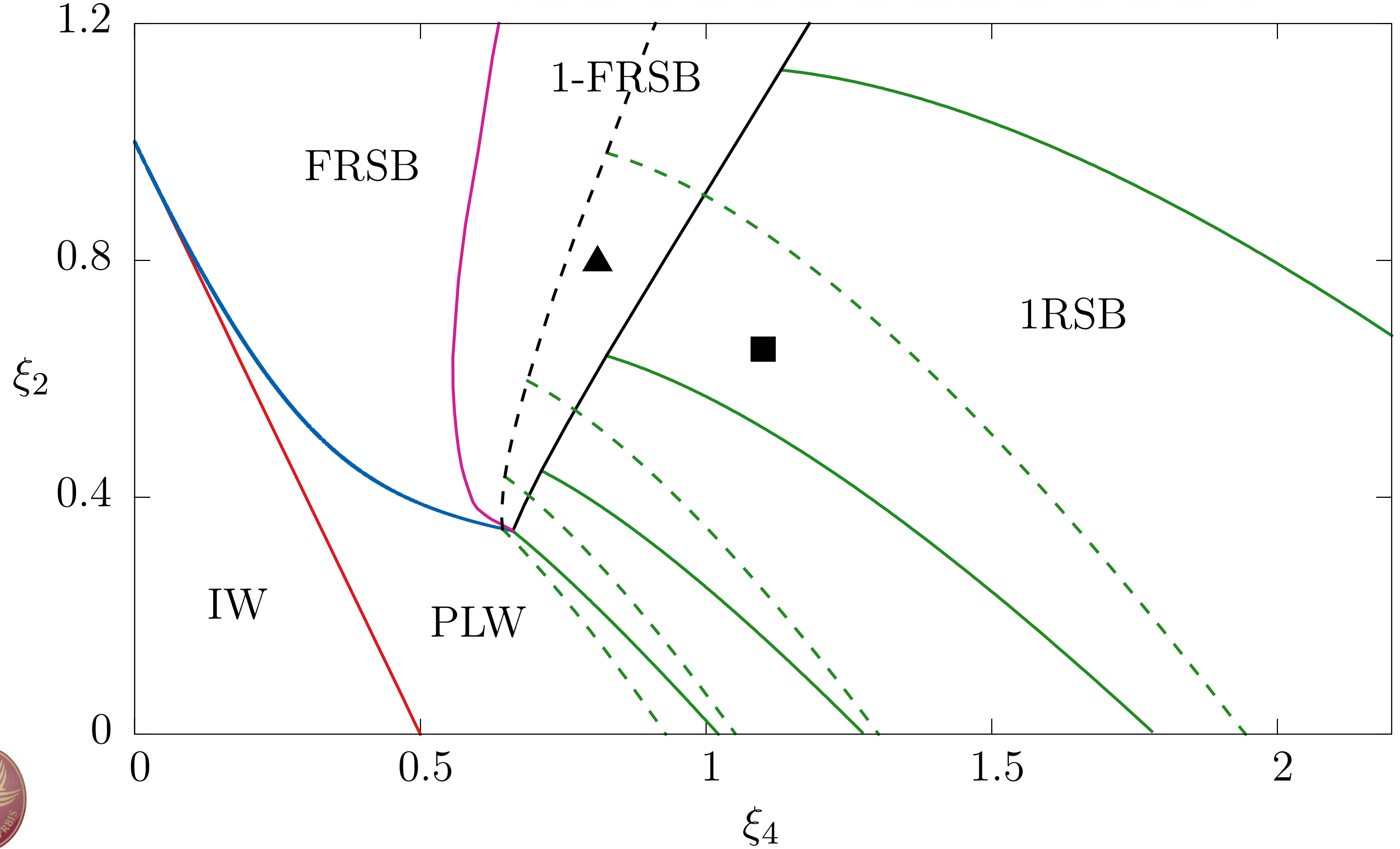
$$\Delta_4 \equiv \frac{|\omega_1 - \omega_2 + \omega_3 - \omega_4|}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}$$

Fully connected 2+4 spherical phasor model

$$\mathcal{H}[\boldsymbol{a}] = -\frac{1}{2}\sum_{n_1 n_2}^{1,N} J_{n_1 n_2} a_{n_1}\ a^\star_{n_2} - \frac{1}{4!}\sum_{n_1 n_2 n_3 n_4}^{1,N} J_{n_1 n_2 n_3 n_4} a_{n_1}\ a^\star_{n_2} a_{n_3} a^\star_{n_4}$$

Phase diagram in "mode-coupling"-like parameters

$$\xi_2 = \frac{\epsilon^2}{4}\beta^2 J_2^2, \quad \xi_4 = \frac{\epsilon^4}{6}\beta^2 J_4^2,$$



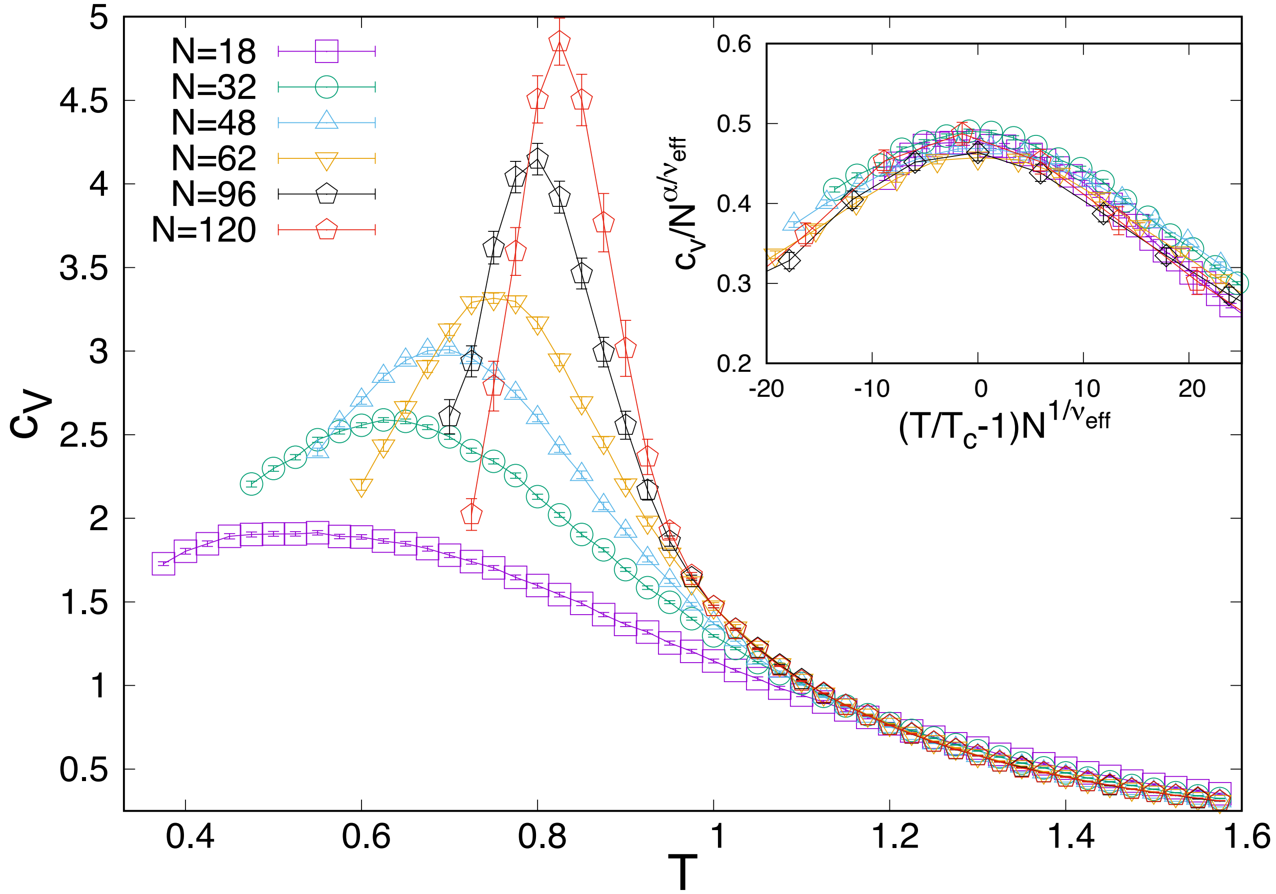
Universality class

$$c_{V_N}(T) = N^{\frac{\alpha}{\nu_{\text{eff}}}} \hat{f}_{C_{V_N}}\left(N^{\frac{1}{\nu_{\text{eff}}}}\ t_N\right)$$

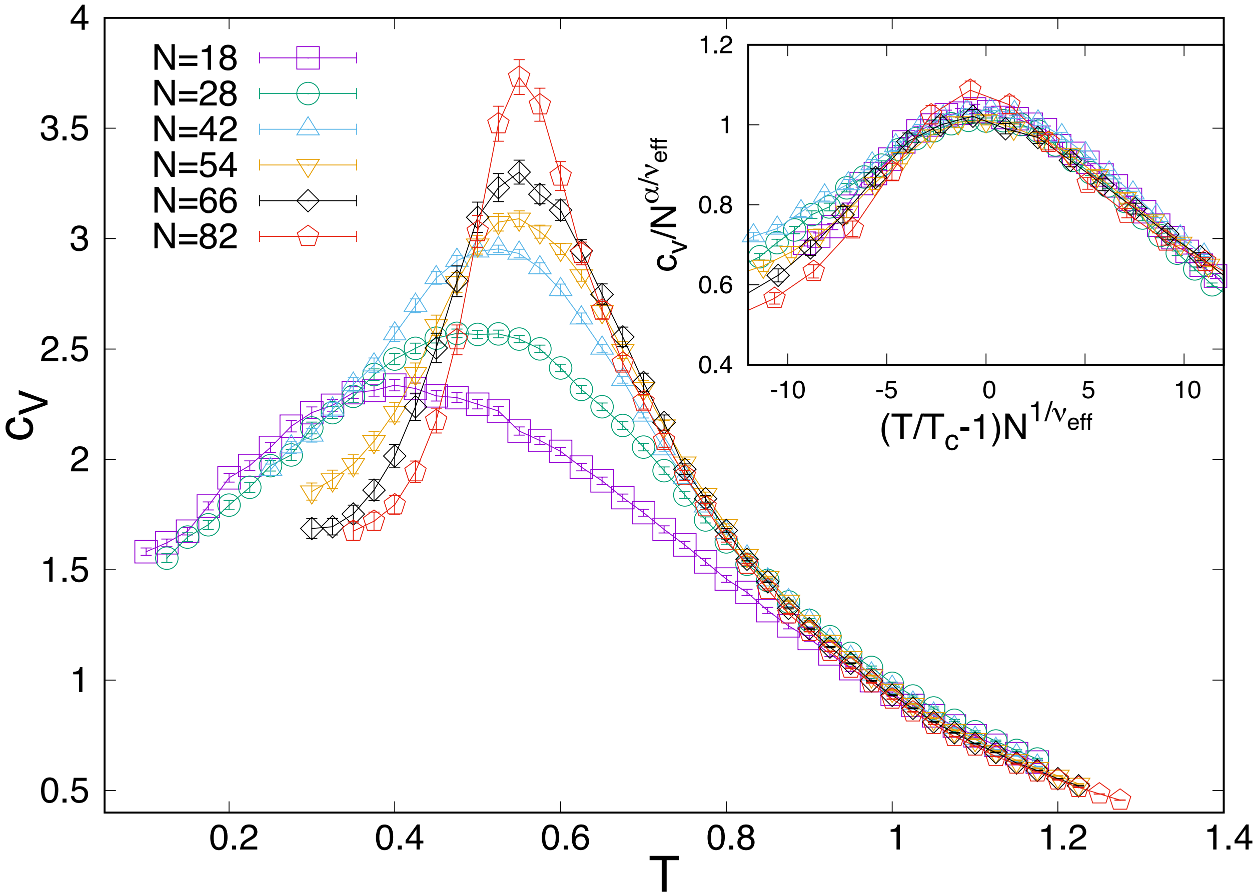
$$\nu_{\text{eff}} = 2\beta + \gamma$$

$1 \leq \nu_{\text{eff}} \leq 2$ for a mean-field model

Free boundary conditions on frequencies



Periodic boundary conditions on frequencies



FBC: $T_c(\infty) = 0.86 \pm 0.03$

PBC: $T_c(\infty) = 0.61 \pm 0.03$

$\alpha = 0.48 \pm 0.05 \qquad 1/\nu_{\text{eff}} = 1.1 \pm 0.1$

PBC: $\alpha = 0.27 \pm 0.05\,, \qquad 1/\nu_{\text{eff}} = 0.86 \pm 0.14$

J. Niedda, G. Gradenigo, LL and G. Parisi, SciPost Phys. 2023

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