Fluctuations of the ground-state energy of spherical spin-glasses

Bertrand Lacroix-A-Chez-Toine, King's College London
In collaboration with Pierre Le Doussal, Yan V. Fyodorov
EPSRC Grant EP/V002473/1

arXiv:2306.11927

Contents

High-dimensional random landscapes

A random landscape $\mathscr{H}(\mathbf{x})$ is a random function of a large number N of degrees of freedom $\mathbf{x} = \{x_1, \cdots, x_N\}$

This is an important topic in physics, mathematics and beyond:

Spin-glass energy landscape

Utility function in economics

Cost function in machine learning

An ubiquitous problem is then to search for the exact (or at least approximate) global minimum or ground state energy of the energy landscape $\mathscr{H}(\mathbf{x})$:

$$e_{\min} = \frac{1}{N} \min_{\mathbf{x}} \mathscr{H}(\mathbf{x})$$

(See e.g. Ros, Fyodorov '22)

Ground-state energy

The intensive ground-state energy (GSE) is self-averaging:

(At least for mean-field models) $\lim_{N\to\infty} e_{\min} = \lim_{N\to\infty} \overline{e_{\min}} = e_{\text{typ}}$

Typical fluctuations extend over a vanishing scale and are described by

$$\lim_{N\to\infty} e_N = 0 \qquad \lim_{N\to\infty} a_N = 0$$

$$\lim_{N\to\infty} \text{Prob}\left[e_{\min} \geq e_{\text{typ}} + e_N + a_N x\right] = \mathscr{P}(-x)$$

$$\lim_{x\to-\infty} \mathscr{P}(x) = 0 \qquad \lim_{x\to+\infty} \mathscr{P}(x) = 1$$

Deriving the limiting distribution is clearly a problem of extreme value statistics for a strongly correlated random process

EVS for independent identically distributed (iid) random variables

This problem is fully characterised for iid random variables:

$$P_{\text{joint}}(x_1, \cdots, x_N) = \prod_{i=1}^{N} p(x_i)$$

$$\lim_{N\to\infty} \text{Prob}\left[x_{\min} \geq x_N + a_N x\right] = \lim_{N\to\infty} \left[\int_{x_N+a_N x}^{\infty} p(u)\, du\right]^N = \mathscr{P}_{\text{i.i.d.}}(-x)$$

The coefficients x_N and a_N Depend explicitly on the parent distribution $p(x)$

$$\lim_{x\to-\infty} \mathscr{P}(x) = 0$$

$$\lim_{x\to+\infty} \mathscr{P}(x) = 1$$

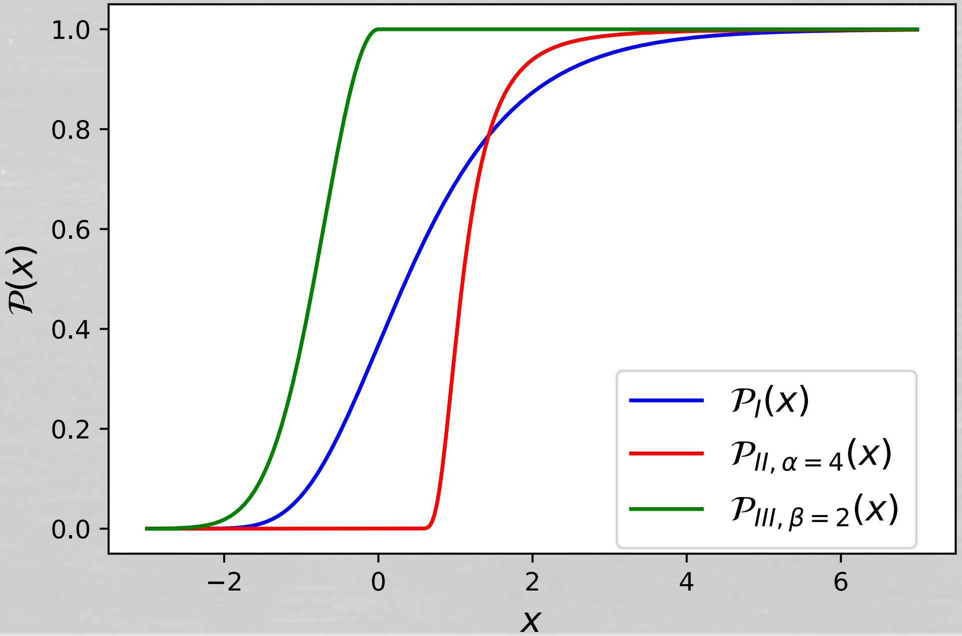
EVS for independent identically distributed (iid) random variables

The distribution of typical fluctuations is universal and falls in one of three universality classes (Fisher-Tippett-Gnedenko theorem)

Gumbel	Fréchet	Weibull
Exponential or faster decay	Power-law decay	Finite edge
$\mathscr{P}_I(x) = \exp(-e^{-x})$	$\mathscr{P}_{II,\alpha}(x) = \exp(-x^{-\alpha})\Theta(x)$	$\mathscr{P}_{III,\beta}(x) = \exp(-x^{\beta})\Theta(-x) + \Theta(x)$

Much more difficult for strongly correlated random variables!

Universal limiting distributions are expected as well but a full characterisation is far from complete!



Contents

- Introduction
- Typical ground-state energy
- Fluctuations of the 2-spin
- Fluctuations for FRSB models
- Conclusion

General Spherical spin-glass model

The general spherical spin-glass model is defined as the Gaussian process

$$\overline{\mathscr{H}(\mathbf{x})} = 0 \qquad \overline{\mathscr{H}(\mathbf{x}_1)\mathscr{H}(\mathbf{x}_2)} = Nf\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{N}\right) \qquad \sum_{i=1}^{N} x_i^2 = N$$

$$f(q) = \sum_{r=2}^{\infty} g_r q^r + \Gamma q = g(q) + \Gamma q \qquad \forall r \geq 2,\ g_r \geq 0$$

The simplest model consists in the p-spin model, where

$$\mathscr{H}(\mathbf{x}) = -\frac{1}{p} \underbrace{\sum_{i_1,\cdots,i_p=1}^{N} J_{i_1,\ldots,i_p} \prod_{j=1}^{p} x_{i_j}}_{\text{Random p-body interaction}} - \underbrace{\sum_{i=1}^{N} h_i x_i}_{\text{Random magnetic field}} \qquad f(q) = \frac{J^2}{p} q^p + \Gamma q$$

$$g_r = \frac{J^2}{p} \delta_{r,p}$$

Random p-body interaction

Random magnetic field

General Spherical spin-glass model

The general spherical spin-glass model is defined as the Gaussian process

$$\overline{\mathscr{H}(\mathbf{x})} = 0 \qquad \overline{\mathscr{H}(\mathbf{x}_1)\mathscr{H}(\mathbf{x}_2)} = Nf\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{N}\right) \qquad \sum_{i=1}^{N} x_i^2 = N$$

$$f(q) = \sum_{r=2}^{\infty} g_r q^r + \Gamma q = g(q) + \Gamma q \qquad \forall r \geq 2,\ g_r \geq 0$$

The energy landscape of this model is:

- A model for the cost function of machine learning algorithms (Choramanska '15)
- A versatile model of constrained optimisation
- A prototypical model of strongly correlated stochastic process

General Spherical spin-glass model

The general spherical spin-glass model is defined as the Gaussian process

$$\overline{\mathscr{H}(\mathbf{x})} = 0 \qquad \overline{\mathscr{H}(\mathbf{x}_1)\mathscr{H}(\mathbf{x}_2)} = Nf\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{N}\right) \qquad \sum_{i=1}^{N} x_i^2 = N$$

$$f(q) = \sum_{r=2}^{\infty} g_r\, q^r + \Gamma q = g(q) + \Gamma q \qquad \forall r \geq 2,\ g_r \geq 0$$

Characterising the typical value and fluctuations of the GSE $e_{\min}$ is thus natural

$$e_{\min} = \frac{1}{N} \min_{\mathbf{x}:\mathbf{x}^2=N} \mathscr{H}(\mathbf{x})$$

General Spherical spin-glass model

The general spherical spin-glass model is defined as the Gaussian process

$$\overline{\mathcal{H}(\mathbf{x})} = 0 \qquad \overline{\mathcal{H}(\mathbf{x}_1)\mathcal{H}(\mathbf{x}_2)} = Nf\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{N}\right) \qquad \sum_{i=1}^{N} x_i^2 = N$$

$$f(q) = \sum_{r=2}^{\infty} g_r q^r + \Gamma q = g(q) + \Gamma q \qquad \forall r \geq 2,\ g_r \geq 0$$

Its typical ground-state energy can be computed using the replica method

$$\lim_{N\to\infty} \frac{1}{N} \ln \overline{Z^n} = \phi_n = \max_{Q>0} \Phi_n(Q) \qquad \Phi_n(Q) = \frac{\beta^2}{2} \sum_{a,b=1}^{n} f(Q_{ab}) + \frac{1}{2} \ln \det Q + \frac{n}{2}\left(1 + \ln 2\pi\right)$$

$$e_{\mathrm{typ}} = \overline{e_{\mathrm{min}}} = -\lim_{\beta\to\infty} \lim_{n\to 0} \frac{\phi_n}{n\beta}$$

General Spherical spin-glass model

The general spherical spin-glass model is defined as the Gaussian process

$$\overline{\mathcal{H}(\mathbf{x})} = 0 \qquad \overline{\mathcal{H}(\mathbf{x}_1)\mathcal{H}(\mathbf{x}_2)} = Nf\left(\frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{N}\right) \qquad \sum_{i=1}^{N} x_i^2 = N$$

$$f(q) = \sum_{r=2}^{\infty} g_r q^r + \Gamma q = g(q) + \Gamma q \qquad \forall r \geq 2,\ g_r \geq 0$$

It yields the Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\min}} = -\min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$$\Psi\left[z(q); v, q_0\right] = \frac{1}{2}\left[v f'(1) + \int_{q_0}^{1} dq\, z(q) f'(q) + \frac{q_0}{v + \int_{q_0}^{1} dq\, z(q)} + \int_{q_0}^{1} \frac{dq}{v + \int_{q}^{1} dr\, z(r)}\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\min}} = -\min_{v\geq 0, 0\leq q_0\leq 1, z(q)} \Psi\left[z(q); v, q_0\right] \qquad \Psi\left[z(q); v, q_0\right] = \frac{1}{2}\left[v f'(1) + \int_{q_0}^{1} dq\, z(q) f'(q) + \frac{q_0}{v + \int_{q_0}^{1} dq\, z(q)} + \int_{q_0}^{1} \frac{dq}{v + \int_{q}^{1} dr\, z(r)}\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\rm typ} = \overline{e_{\rm min}} = -\min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma > \Gamma_{\rm RSB} = g''(1) - g'(1)$ the solution is RS: $q_0 = 1$ and/or $z(q) = 0$

$$e_{\rm typ} = \overline{e_{\rm min}} = -\frac{1}{2}\min_{v \geq 0}\left[v\, f'(1) + \frac{1}{v}\right] = -\sqrt{g'(1) + \Gamma}$$

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma > \Gamma_{\text{RSB}} = g''(1) - g'(1)$ the solution is RS: $q_0 = 1$ and/or $z(q) = 0$

$$e_{\text{typ}} = \overline{e_{\min}} = -\frac{1}{2} \min_{v \geq 0} \left[v f'(1) + \frac{1}{v} \right] = -\sqrt{g'(1) + \Gamma}$$

For the 2-spin $g(q) = \frac{J^2}{2} q^2$ one has $\Gamma_{\text{RSB}} = 0$

The typical GSE is always RS

$$e_{\text{typ}} = -\sqrt{J^2 + \Gamma}$$

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

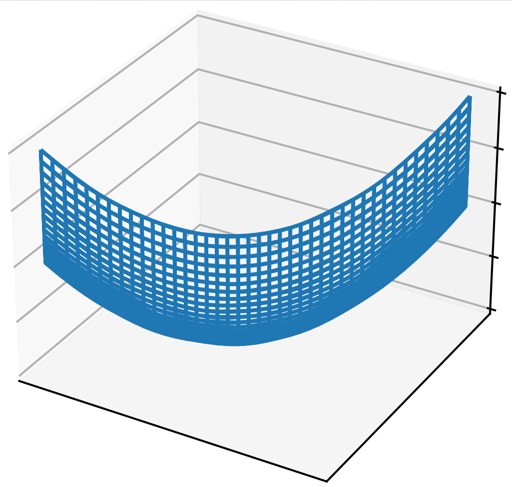
$$e_{\rm typ} = \overline{e_{\rm min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma > \Gamma_{\rm RSB} = g''(1) - g'(1)$ the solution is RS: $q_0 = 1$ and/or $z(q) = 0$
- The energy landscape $\mathscr{H}(\mathbf{x})$ is "topologically trivial" for this type of models:

  it displays a sub-exponential number of local minima



Replica symmetry (breaking) of the solution

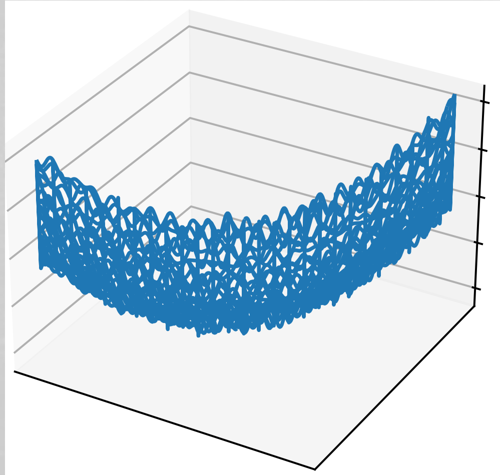
Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi \left[z(q); v, q_0 \right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\text{RSB}}$ the solution is RSB
- The energy landscape $\mathscr{H}(\mathbf{x})$ is "topologically complex" for this type of models: it displays an exponential number of local minima



Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{typ} = \overline{e_{min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi \left[z(q); v, q_0 \right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{RSB}$ the solution is RSB
- Its number of RSB depends on the sign of the Schwarzian derivative

$$\mathcal{S}[g'(q)] = \frac{g^{(4)}(q)}{g''(q)} - \frac{3}{2} \left(\frac{g^{(3)}(q)}{g''(q)} \right)^2$$

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

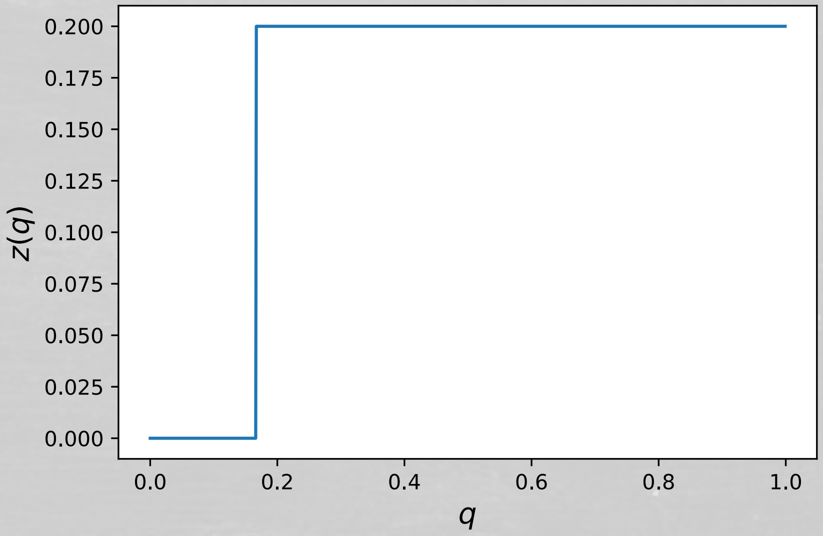
$$e_{\text{typ}} = \overline{e_{\min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi \left[z(q); v, q_0 \right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\text{RSB}}$ the solution is RSB
- If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] < 0$ the solution is 1RSB

$$z(q) = \begin{cases} 0 \, , & q < q_0 \\ m_0 > 0 \, , & q > q_0 \end{cases}$$



The $p > 2$-spin is 1RSB

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\text{min}}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\text{RSB}}$ the solution is RSB
- If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] < 0$ the solution is 1RSB

The exponentially many local minima of the random energy landscape are isolated, separated by high barriers

Only local minima are found in a small range of energy around e_{min}

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

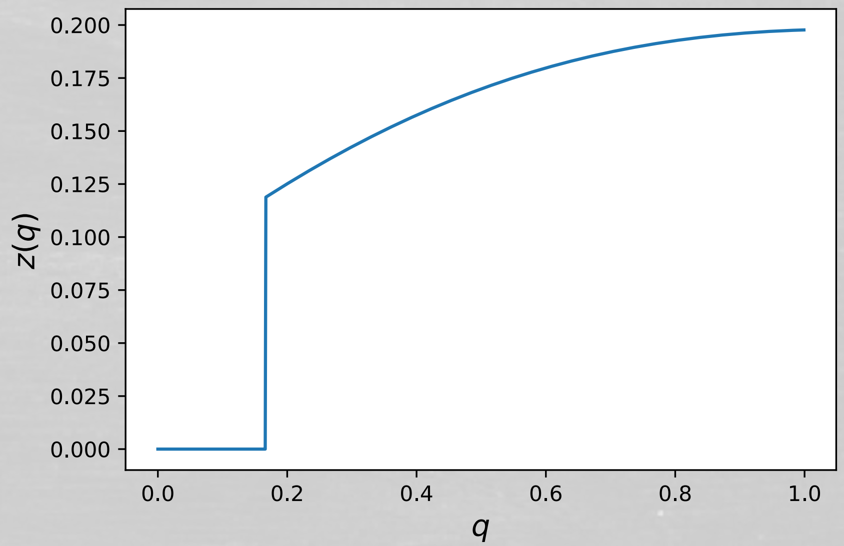
$$e_{\rm typ} = \overline{e_{\min}} = - \min_{v\geq 0, 0\leq q_0 \leq 1, z(q)} \Psi\left[z(q); v, q_0\right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\rm RSB}$ the solution is RSB
- If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] > 0$ the solution is FRSB

$$z(q) = \begin{cases} 0\ , \ \ q < q_0 \\ \dfrac{g^{(3)}(q)}{2g''(q)^{3/2}} \geq 0\ , \ \ q > q_0 \end{cases}$$



Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\text{typ}} = \overline{e_{\min}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi \left[z(q); v, q_0 \right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\text{RSB}}$ the solution is RSB
- If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] > 0$ the solution is FRSB

There are many flat directions of the landscape

All types of saddles are found in a small range of energy around $e_{\min}$

Replica symmetry (breaking) of the solution

Crisanti-Sommers formula (Crisanti & Sommers '92)

$$e_{\mathrm{typ}} = \overline{e_{\mathrm{min}}} = - \min_{v \geq 0, 0 \leq q_0 \leq 1, z(q)} \Psi \left[z(q); v, q_0 \right]$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = 0$

The explicit form of $z(q)$ depends on the covariance function $f(q) = g(q) + \Gamma q$:

- For $\Gamma < \Gamma_{\mathrm{RSB}}$ the solution is RSB
- If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] > 0$ the solution is FRSB

$$e_{\mathrm{typ}} = - \left(q_{\mathrm{typ}} \sqrt{g''(q_{\mathrm{typ}})} + \int_{q_{\mathrm{typ}}}^{1} dq \sqrt{g''(q)} \right) \qquad \Gamma = q_{\mathrm{typ}} g''(q_{\mathrm{typ}}) - g'(q_{\mathrm{typ}})$$

As $\Gamma \to \Gamma_{\mathrm{RSB}} = g''(1) - g'(1)$, $q_{\mathrm{typ}} \to 1$ and the RS solution is recovered

Contents

Spherical 2-spin model

For the spherical 2-spin model

$$\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i \qquad \sum_{i=1}^{N} x_i^2 = N$$

Random 2-body interaction — Random magnetic field

$$\overline{J_{ij}} = 0 \qquad \overline{J_{ij}J_{kl}} = \frac{J^2}{N}(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\overline{h_i} = 0 \qquad \overline{h_i h_j} = \Gamma \delta_{ij}$$

Constrained optimisation problem:

$$e_{\min} = \frac{1}{N} \min_{\mathbf{x}:\mathbf{x}^2=N} \mathscr{H}(\mathbf{x})$$

Studied in detail in computer science (Conn et al. '00, Tisseur & Meerberger '01, ...)

Spherical 2-spin model

For the spherical 2-spin model $\mathcal{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\quad \sum_{i=1}^{N} x_i^2 = N$

In absence of magnetic field $\Gamma = 0$:

$$e_{\min} = \frac{\lambda_{\min}}{2}$$

$\lambda_{\min}$: Lowest eigenvalue of GOE matrix

$e_{\mathrm{typ}} = -J$

Distribution of the ground-state energy (GSE):

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)}$$

Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\quad \sum_{i=1}^{N} x_i^2 = N$

In absence of magnetic field $\Gamma = 0$:

$e_{min} = \frac{\lambda_{min}}{2}$ $\quad \lambda_{min}$: Lowest eigenvalue of GOE matrix $\quad e_{typ} = -J$

Three non-trivial regimes of fluctuations

$$P_N(e) \approx \begin{cases} e^{-N\mathscr{L}(e)} & \text{Left atypical fluctuations } e < e_{typ} \\ 2N^{2/3}\mathscr{F}_1\left(-2N^{2/3}(e - e_{typ})\right) & \text{Typical fluctuations } N^{2/3}|e - e_{typ}| = O(1) \\ e^{-N^2\mathscr{R}(e)} & \text{Right atypical fluctuations } e > e_{typ} \end{cases}$$

Ben Arous, Dembo & Guionnet '01
Majumdar & Vergassola '09

Tracy & Widom '96

Dean, Majumdar '06

$\mathscr{F}_1(x)$: Tracy-Widom GOE distribution

GOE: Gaussian Orthogonal Ensemble

Matching of the tails

One can show explicitly a matching between the tails of the TW and the behaviours of the large deviation functions (LDFs)

Typical fluctuations: Tracy-Widom

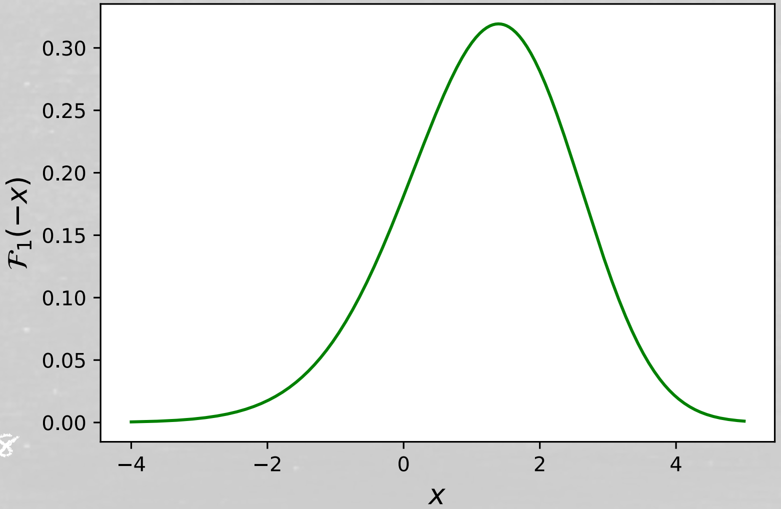
$$-\ln \mathcal{F}_1(-x) = \begin{cases} \dfrac{2}{3}|x|^{3/2}, & x \to -\infty \\ \dfrac{|x|^3}{24}, & x \to +\infty \end{cases}$$

Atypical fluctuations: LDFs

$$N\mathcal{L}(e) \approx \frac{2N}{3}\left(-2(e+J)\right)^{3/2}, \; e \to -J_-$$

$$N^2\mathcal{R}(e) \approx \frac{N^2}{24}\left(2(e+J)\right)^{3}, \; e \to -J_+$$

$$x = 2N^{2/3}(e+J)$$



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

For positive magnetic field $\Gamma > 0$:

The ground-state energy satisfies a central limit theorem (Chen & Sen '17)

Gaussian typical fluctuations

$$e_{\text{typ}} = \overline{e_{\text{min}}} = -\lim_{\beta\to\infty}\lim_{n\to 0}\frac{\phi_n}{n\beta} = -\sqrt{J^2 + \Gamma}$$

$$\lim_{N\to\infty} N \operatorname{Var}(e_{\text{min}}) = \mathscr{V}_{\text{min}} = \frac{\Gamma}{2}$$

Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\quad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

The atypical fluctuations are described by a large deviation function (LDF)

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e)$$

Spherical 2-spin model

For the spherical 2-spin model $\mathcal{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

The atypical fluctuations are described by a large deviation function (LDF)

$$\mathcal{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e)$$

Its Legendre transform is the scaled cumulant generating function (CGF) and can be computed using replica computations

$$\lim_{\beta\to\infty} \frac{1}{N} \ln \overline{Z^{s/\beta}} = \frac{1}{N} \ln \overline{e^{-Nse_{\min}}} = \frac{1}{N} \ln \int de\, e^{-N[se + \mathcal{L}(e)]}$$

$$\phi(s) = -\min_e \left[se + \mathcal{L}(e)\right] = \lim_{\beta\to\infty} \phi_{s/\beta}$$

$$e_{\min} = -\lim_{\beta\to\infty} \frac{1}{N\beta} \ln Z$$

$$\lim_{N\to\infty} \frac{1}{N} \ln \overline{Z^n} = \phi_n$$

$$Q_{ab} = \frac{\mathbf{x}_a \cdot \mathbf{x}_b}{N}$$

Overlap matrix

Spherical 2-spin model

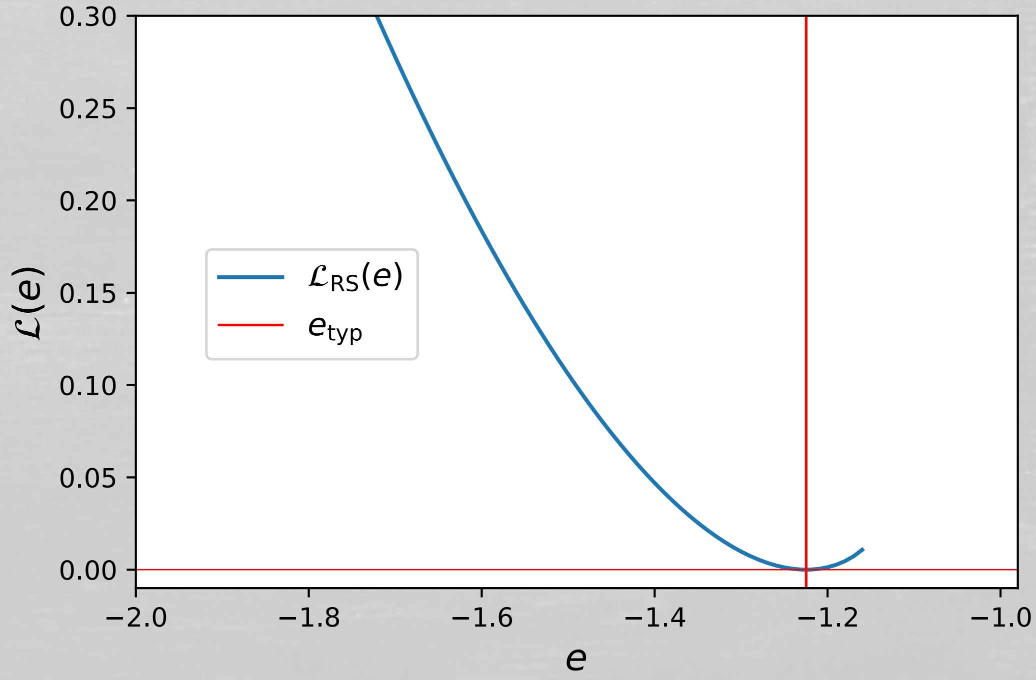
For the spherical 2-spin model $\quad \mathcal{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i \qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

The CGF and the LDF were first computed using a RS ansatz

Its expression extends for $e < e_{c,\mathrm{RS}}$

RS: Fyodorov & Le Doussal '14



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\quad \sum_{i=1}^{N} x_i^2 = N$

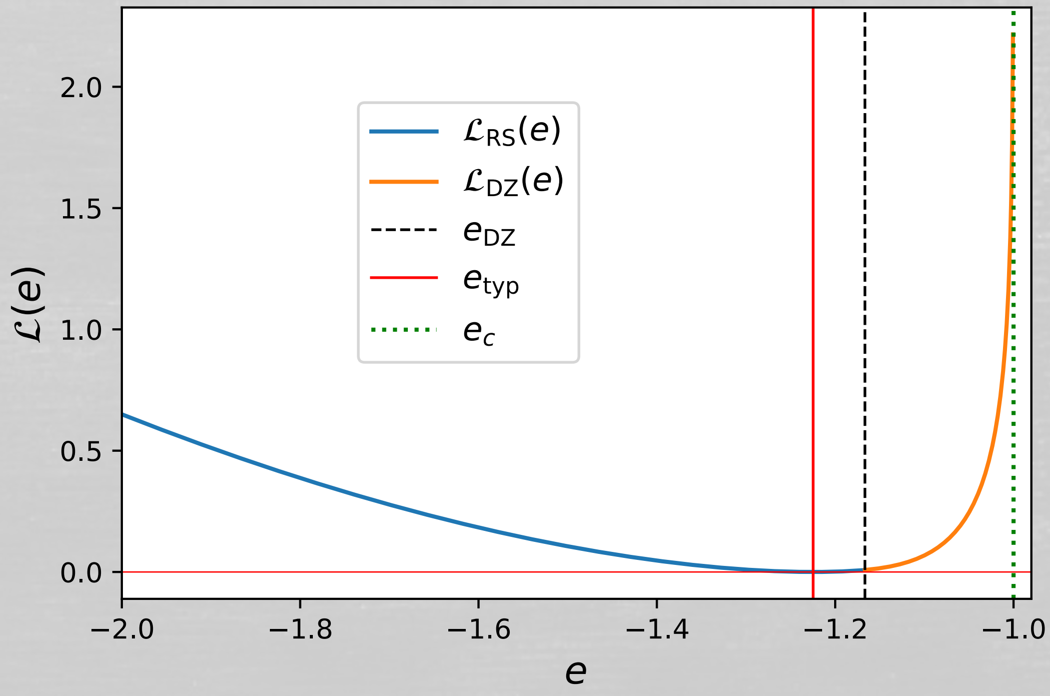
$\overline{h_i h_j} = \Gamma \delta_{ij}$

Using a rigorous approach the CGF and the LDF were shown to display two branches:

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e) = \begin{cases} \mathscr{L}_{\mathrm{RS}}(e)\,, & e < e_{\mathrm{DZ}} \\ \mathscr{L}_{\mathrm{DZ}}(e)\,, & e_c > e > e_{\mathrm{DZ}}\,, \\ +\infty\,, & e > e_c \end{cases}$$

RS: Fyodorov & Le Doussal '14

Rigorous: Dembo & Zeitouni '15



The real extent of the RS solution is $e < e_{\mathrm{RSB}} \leq e_{c,\mathrm{RS}}$

The transition is of third order

$$\mathscr{L}_{\mathrm{RS}}(e) - \mathscr{L}_{\mathrm{RSB}}(e) \propto (e - e_{\mathrm{RSB}})^3,\ e \to e_{\mathrm{RSB}}$$

Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

We showed that the mechanism behind these two branches is RSB

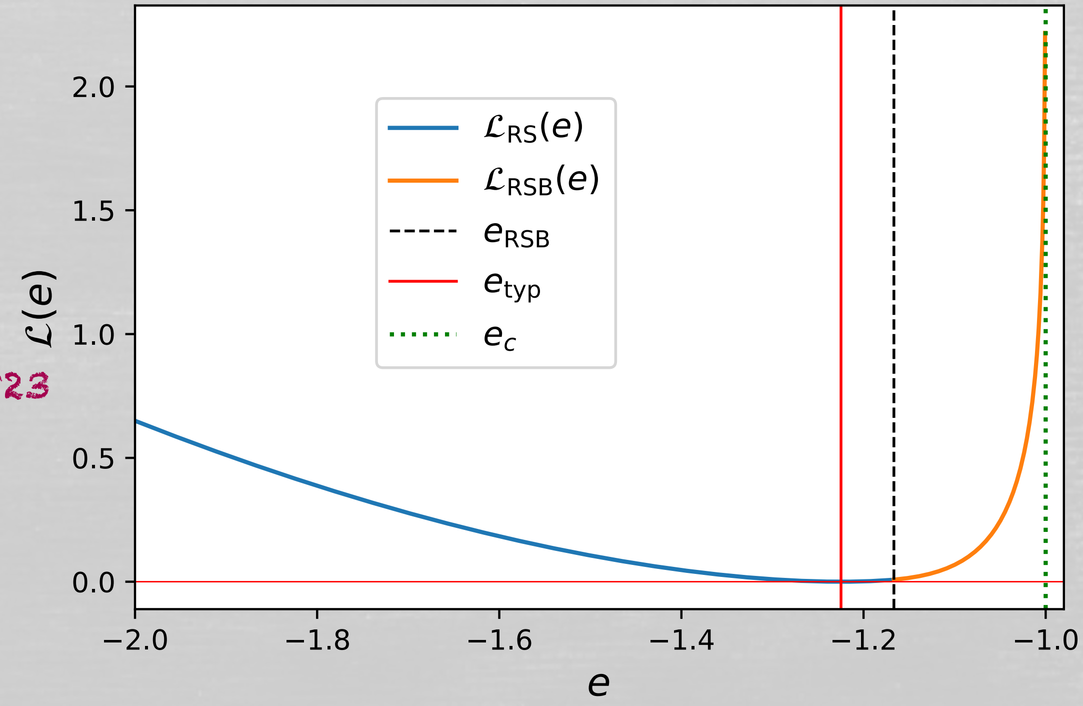
$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e) = \begin{cases} \mathscr{L}_{\mathrm{RS}}(e)\,, & e < e_{\mathrm{RSB}} \\ \mathscr{L}_{\mathrm{RSB}}(e)\,, & e_c > e > e_{\mathrm{RSB}}\,, \\ +\infty\,, & e > e_c \end{cases}$$

RS: Fyodorov & Le Doussal '14

Rigorous: Dembo & Zeitouni '15

Mechanism:

LACT, Fyodorov & Le Doussal '23



One eigenvalues of $A_{(ab),(cd)} = \dfrac{\delta^2 \Phi_{n=s/\beta}(Q)}{\delta Q_{ab} \delta Q_{cd}}$

becomes positive for $e > e_{\mathrm{RSB}}$

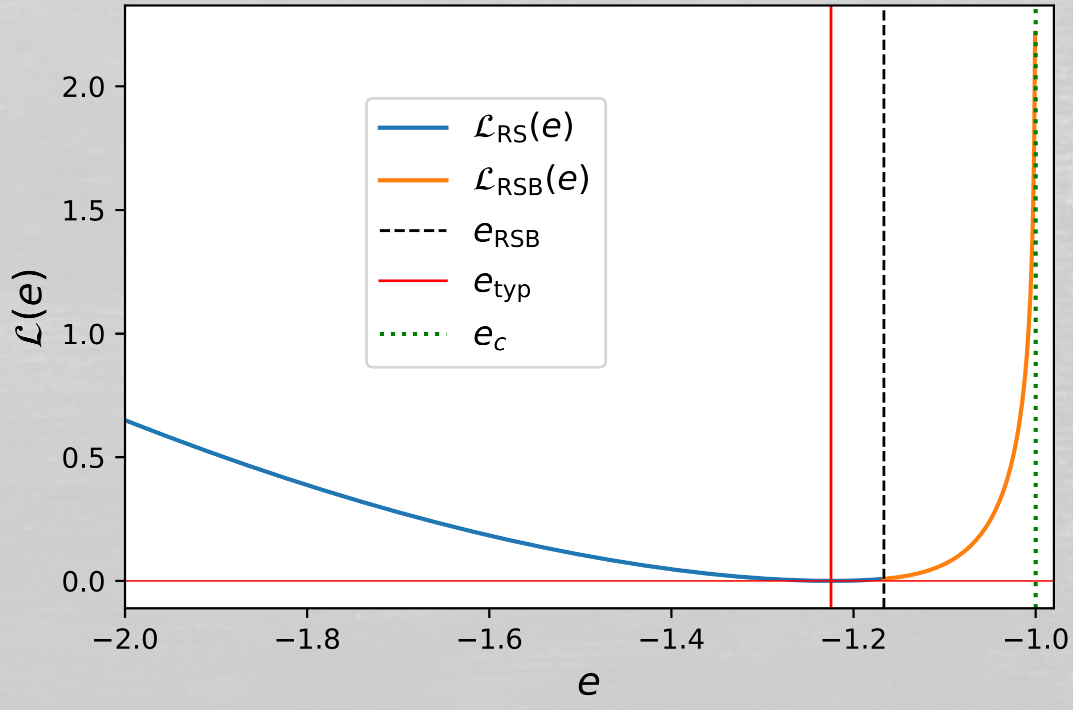
Spherical 2-spin model

For the spherical 2-spin model $\mathcal{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i \qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

The CGF and the LDF each display two distinct branches: an RS and an RSB branch

$$\mathcal{L}(e) = -\lim_{N\to\infty}\frac{1}{N}\ln P_N(e) = \begin{cases} \mathcal{L}_{\mathrm{RS}}(e)\,, & e < e_{\mathrm{RSB}} \\ \mathcal{L}_{\mathrm{RSB}}(e)\,, & e_c > e > e_{\mathrm{RSB}}\,, \\ +\infty\,, & e > e_c \end{cases}$$



The LDF diverges beyond a finite critical energy $e_c = e_{\mathrm{typ}}(\Gamma = 0) = -J$

$$\mathcal{L}_{\mathrm{RSB}}(e) \approx -\frac{1}{2}\ln(e_c - e)\,,\ \ e \to e_c$$

Spherical 2-spin model

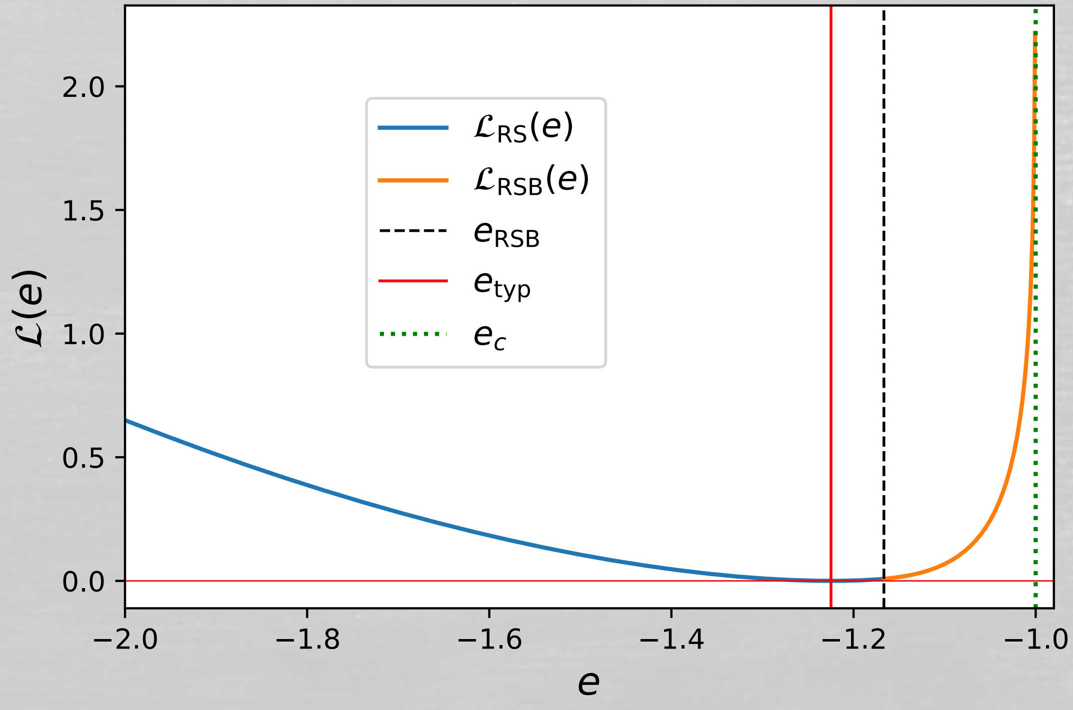
For the spherical 2-spin model $\qquad \mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i \qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

The CGF and the LDF each display two distinct branches: an RS and an RSB branch

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e) = \begin{cases} \mathscr{L}_{\rm RS}(e)\,, & e < e_{\rm RSB} \\ \mathscr{L}_{\rm RSB}(e)\,, & e_c > e > e_{\rm RSB}\,, \\ +\infty\,, & e > e_c \end{cases}$$

The fluctuations for $e > e_c$ are either described by a LDF with rate $s_N \gg N$ (Most probably N^2) or completely suppressed



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\qquad \sum_{i=1}^{N} x_i^2 = N$

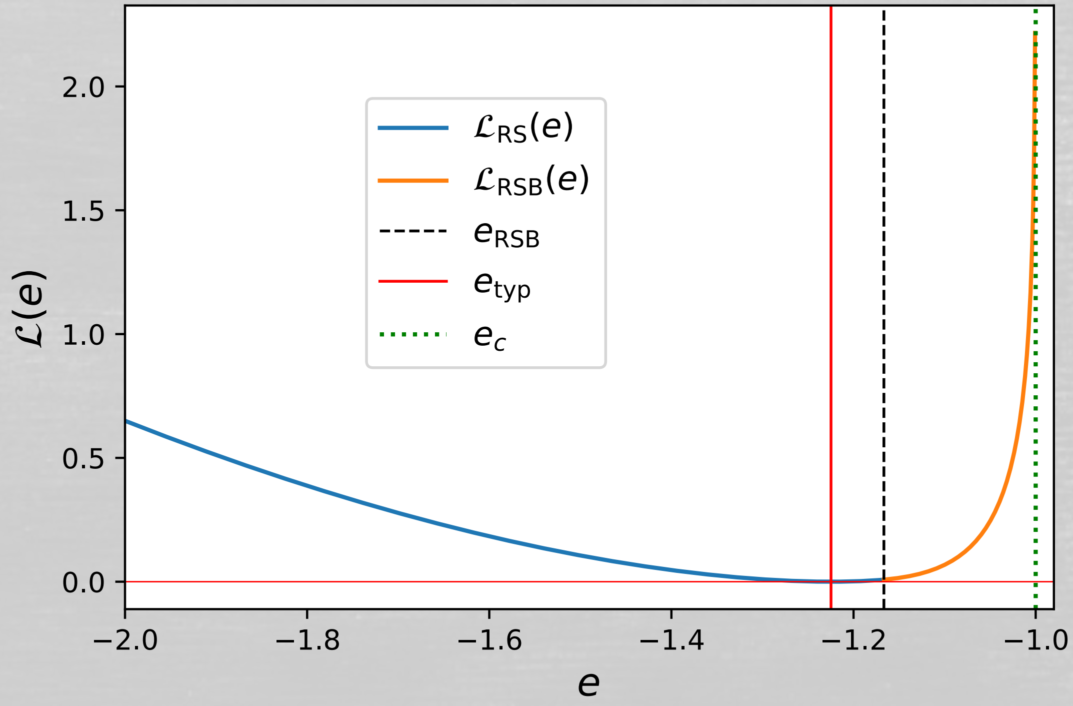
$\overline{h_i h_j} = \Gamma \delta_{ij}$

The CGF and the LDF each display two distinct branches: an RS and an RSB branch

$$\mathscr{L}(e) = -\lim_{N\to\infty}\frac{1}{N}\ln P_N(e) = \begin{cases} \mathscr{L}_{\mathrm{RS}}(e)\,, & e < e_{\mathrm{RSB}} \\ \mathscr{L}_{\mathrm{RSB}}(e)\,, & e_c > e > e_{\mathrm{RSB}}\,, \\ +\infty\,, & e > e_c \end{cases}$$

The RS branch of the LDF matches in the vicinity of e_{typ} the Gaussian tail of the limiting PDF

$$\mathscr{L}_{\mathrm{RS}}(e) \approx \frac{\left(e - e_{\mathrm{typ}}\right)^2}{2\mathscr{V}_{\min}}, \; e \to e_{\mathrm{typ}}$$



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\qquad \sum_{i=1}^{N} x_i^2 = N$

$\overline{h_i h_j} = \Gamma \delta_{ij}$

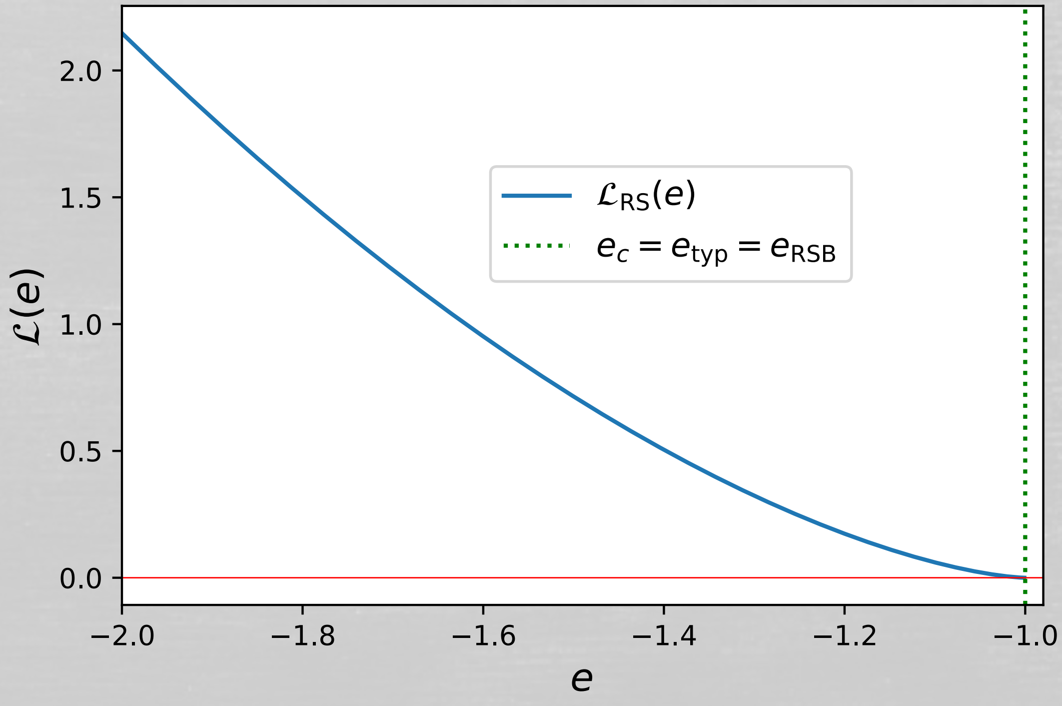
Taking the limit $\Gamma \to 0$, only the RS branch appears

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e) = \begin{cases} \mathscr{L}_{\mathrm{RS}}(e)\,, & e < e_{\mathrm{RSB}} = -J \\ +\infty\,, & e > e_c = -J \end{cases}$$

The rescaled variance vanishes $\mathscr{V}_{\min} \to 0$

The 3/2 tail of the TW is recovered

$$\mathscr{L}_{\mathrm{RS}}(e) \approx \frac{4\sqrt{2}}{3} |e+J|^{3/2},\; e \to e_{\mathrm{typ}}(\Gamma=0) = e_c = -J$$



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\quad \sum_{i=1}^{N} x_i^2 = N$

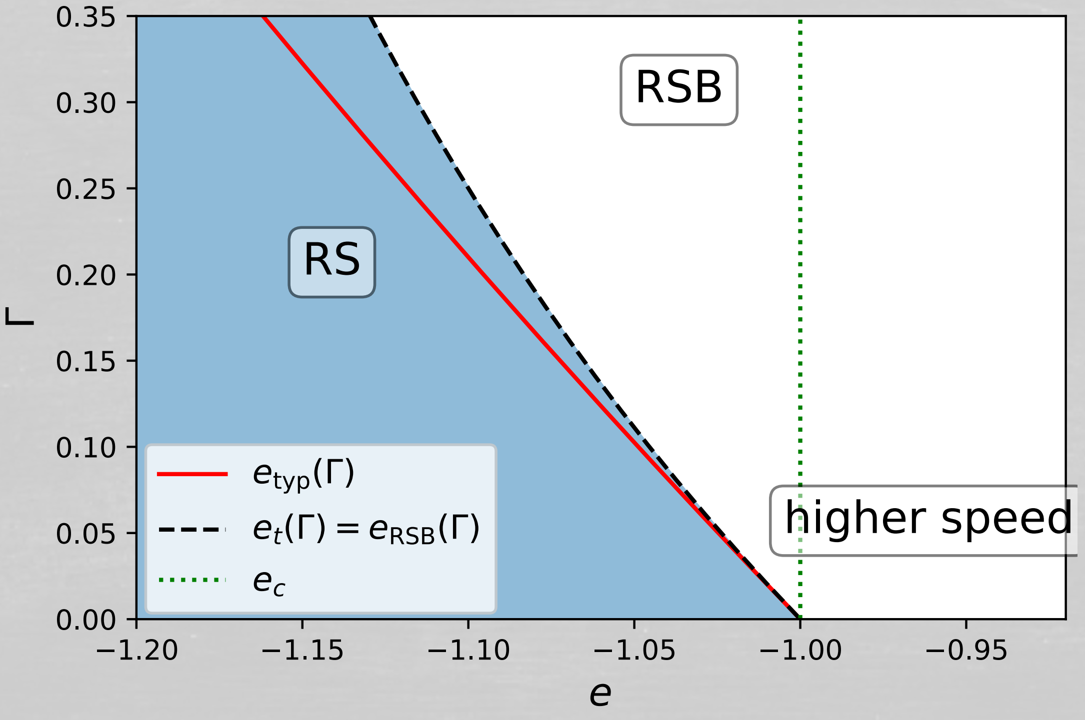
$\overline{h_i h_j} = \Gamma \delta_{ij}$

The LDF of the 2-spin are described by the following phase diagram

$$e_c = -J$$

$$e_{\text{typ}}(\Gamma) = -\sqrt{J + \Gamma^2}$$

$$e_{\text{RSB}}(\Gamma) = -J\left(1 + \frac{J^2}{2(J^2 + \Gamma)}\right)$$



Spherical 2-spin model

For the spherical 2-spin model $\mathscr{H}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j} x_i J_{ij} x_j - \sum_{i=1} h_i x_i$ $\sum_{i=1}^{N} x_i^2 = N$

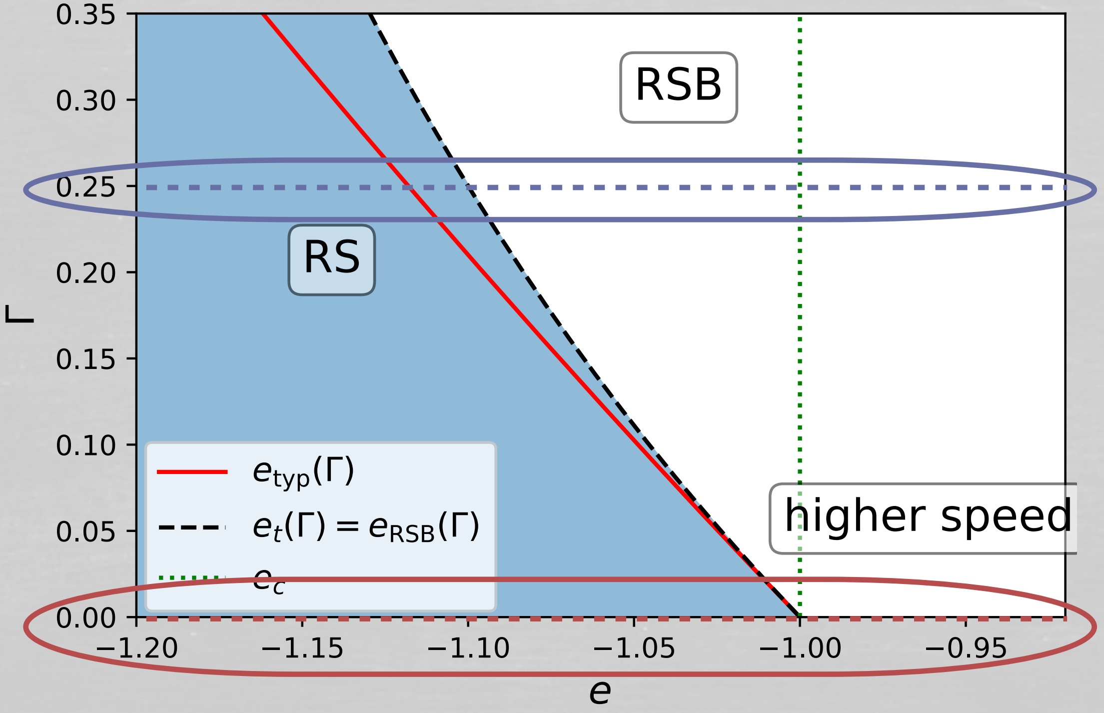
$\overline{h_i h_j} = \Gamma \delta_{ij}$

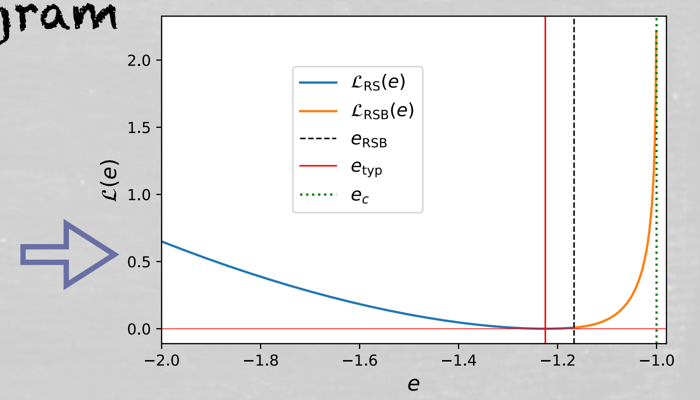
The LDF of the 2-spin are described by the following phase diagram

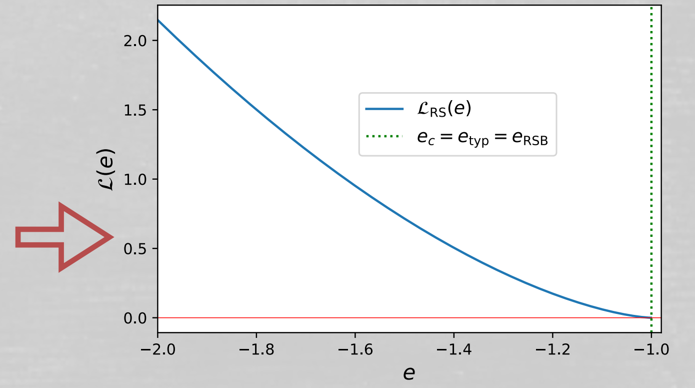
$e_c = -J$

$e_{typ}(\Gamma) = -\sqrt{J + \Gamma^2}$

$e_{RSB}(\Gamma) = -J\left(1 + \frac{J^2}{2(J^2 + \Gamma)}\right)$







Contents

Typical fluctuations of the GSE

The distribution of the GSE

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)} \approx a_N \mathscr{P}\left(a_N(e - e_{\text{typ}} - e_N)\right) \qquad N \to \infty$$

Typical fluctuations of the GSE

The distribution of the GSE

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)} \approx a_N \mathscr{P}\left(a_N(e - e_{\text{typ}} - e_N)\right) \qquad N \to \infty$$

- For $\Gamma > 0$, the GSE satisfies a central limit theorem and for models with FRSB

$$e_{\text{typ}} = -\left(q_{\text{typ}}\sqrt{g''(q_{\text{typ}})} + \int_{q_{\text{typ}}}^{1} dq\,\sqrt{g''(q)}\right)$$

(Chen & Sen '17)

$$\Gamma = q_{\text{typ}} g''(q_{\text{typ}}) - g'(q_{\text{typ}})$$

$$\lim_{N\to\infty} N\,\text{Var}(e_{\min}) = \mathscr{V}_{\min} = g(q_{\text{typ}}) + \frac{\Gamma - g'(q_{\text{typ}})}{2}$$

Typical fluctuations of the GSE

The distribution of the GSE

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)} \approx a_N \mathscr{P}\left(a_N(e - e_{\text{typ}} - e_N)\right) \qquad N \to \infty$$

- For $\Gamma = 0$ and the $p > 2$-spin model:

The typical fluctuations of the GSE are described by a Gumbel distribution

$$e_N \sim \frac{1}{2N} \ln N$$

$$a_N \sim \frac{1}{N}$$

(Subag & Zeitouni '17)

$$\lim_{N\to\infty} \text{Prob}\left[e_{\min} \geq e_{\text{typ}} + e_N + a_N x\right] = \mathscr{G}(-x)$$

$$\mathscr{G}(x) = \exp\left(-e^{-x}\right)$$

Typical fluctuations of the GSE

The distribution of the GSE

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)} \approx a_N \mathscr{P}\left(a_N(e - e_{\text{typ}} - e_N)\right) \qquad N \to \infty$$

- No general result for $\Gamma = 0$
- There exists a general bound from the average density of minima of $\mathscr{H}(\mathbf{x})$:

$$P_N(e) = \overline{\delta\left(e - e_{\min}\right)} \leq \overline{\mathscr{N}_{\min}(e)} = \sum_{\alpha:\text{minima of } \mathscr{H}(\mathbf{x})} \overline{\delta(e - e_\alpha)}$$

The density of minima can be computed using Kac-Rice formula

$$\overline{\mathscr{N}_{\min}(e)} = \int_{\mathbf{x}^2 = N} d\mathbf{x}\, \overline{\delta\left(e - \frac{\mathscr{H}(\mathbf{x})}{N}\right) \delta\left(\nabla \mathscr{H}(\mathbf{x})\right) \det\left(\nabla^2 \mathscr{H}(\mathbf{x})\right) \Theta(\nabla^2 \mathscr{H}(\mathbf{x}))}$$

(See e.g. Ros, Fyodorov '22)

Atypical fluctuations of the GSE

An alternative indirect method consists in analysing the atypical fluctuations

They are characterised by the LDF:

$$\mathscr{L}(e) = - \lim_{N\to\infty} \frac{1}{N} \ln P_N(e)$$

The behaviour of the LDF in the vicinity of e_{typ}

$$N\mathscr{L}(e) \approx N\beta\,|e - e_{\text{typ}}|^{\alpha}\,, \quad e \to e_{\text{typ}}$$

Is expected to match the left tail of the PDF

$$-\ln \mathscr{P}(a_N x) = a_N^{\alpha}\beta\,|x|^{\alpha}\,, \quad x \to -\infty$$

$$a_N \sim N^{1/\alpha}$$

Question investigated for Sherrington-Kirkpatrick (SK) in a series of papers by Parisi & Rizzo '08 '09 '10

Atypical fluctuations of the GSE

An alternative indirect method consists in analysing the atypical fluctuations

They are characterised by the LDF:

$$\mathcal{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e) \leq -\Sigma_{\min}(e) = \lim_{N\to\infty} -\frac{1}{N} \ln \overline{\mathcal{N}_{\min}(e)}$$

The LDF is bounded in terms of the annealed complexity of minima

Atypical fluctuations of the GSE

The atypical fluctuations are characterised by the LDF:

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln P_N(e)$$

As for the 2-spin model, the scaled CGF can be computed using replica computations

$$\phi(s) = -\min_{e} [se + \mathscr{L}(e)] = \lim_{\beta\to\infty} \phi_{s/\beta}$$

$$e_{\min} = -\lim_{\beta\to\infty} \frac{1}{N\beta} \ln Z$$

$$\lim_{N\to\infty} \frac{1}{N} \ln \overline{Z^n} = \phi_n = \max_{Q>0} \Phi_n(Q)$$

$$\Phi_n(Q) = \frac{\beta^2}{2} \sum_{a,b=1}^{n} f(Q_{ab}) + \frac{1}{2} \ln \det Q + \frac{n}{2} \left(1 + \ln 2\pi\right)$$

Atypical fluctuations of the GSE

The expression of the CGF takes a similar form as the Crisanti-Sommers formula

$$\phi(s) = -\min_e \left[se + \mathcal{L}(e)\right] = \lim_{\beta\to\infty} \phi_{s/\beta} = \begin{cases} \max\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \; s > 0 \\ 0 & , \; s = 0 \\ \min\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \; s < 0 \end{cases}$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = s$

$$\Phi(s) = \frac{s}{2}\left[sf(q_0) + vf'(1) + \int_{q_0}^1 dq\, z(q) f'(q)\right] + \frac{1}{2}\left[\ln\left(sq_0 + v + \int_{q_0}^1 dq\, z(q)\right) - \ln\left(v + \int_{q_0}^1 dq\, z(q)\right)\right]$$

$$+ \frac{s}{2}\int_{q_0}^1 \frac{dq}{v + \int_q^1 dr\, z(r)} .$$

(LACT, Fyodorov & Le Doussal '23)

Atypical fluctuations of the GSE

The expression of the CGF takes a similar form as the Crisanti-Sommers formula

$$\phi(s) = -\min_{e}\left[se + \mathscr{L}(e)\right] = \lim_{\beta\to\infty}\phi_{s/\beta} = \begin{cases} \max\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \ s > 0 \\ 0 & , \ s = 0 \\ \min\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \ s < 0 \end{cases}$$

$z(q)$: non-decreasing function of $q \in [0,1]$ with $z(q < q_0) = s$

As for the 2-spin, the CGF may undergo RSB transitions
The location of the transition depends on s

(LACT, Fyodorov & Le Doussal '23)

Atypical fluctuations of the GSE

The expression of the CGF takes a similar form as the Crisanti-Sommers formula

$$\phi(s) = -\min_e \left[se + \mathscr{L}(e)\right] = \lim_{\beta\to\infty} \phi_{s/\beta} = \begin{cases} \max\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \; s > 0 \\ 0 & , \; s = 0 \\ \min\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \; s < 0 \end{cases}$$

The typical GSE is recovered

$$e_{\text{typ}} = -\phi'(0) = -\min_{v\geq 0, 0<q_0<1, z(q)} \Phi'(0)$$

And the rescaled variance can be obtained

$$\lim_{N\to\infty} N \, \text{Var}(e_{\min}) = \mathscr{V}_{\min}$$

(LACT, Fyodorov & Le Doussal '23)

Atypical fluctuations of the GSE

The expression of the CGF takes a similar form as the Crisanti-Sommers formula

$$\phi(s) = -\min_e \left[se + \mathcal{L}(e) \right] = \lim_{\beta\to\infty} \phi_{s/\beta} = \begin{cases} \max\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \ s > 0 \\ 0 & , \ s = 0 \\ \min\limits_{v\geq 0, 0<q_0<1, z(q)} \Phi(s) & , \ s < 0 \end{cases}$$

Taking the Legendre transform, one obtains the LDF

$$\mathcal{L}(e) = -\min_s \left[se + \phi(s) \right]$$

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF

For a model with FRSB phase, one obtains for the LDF

$$\mathscr{L}(e) = -\min_s \left[se + \phi(s) \right]$$

The criterion for the number of RSB is again Schwarzian derivative

$$\mathcal{S}[g'(q)] = \frac{g^{(4)}(q)}{g''(q)} - \frac{3}{2}\left(\frac{g^{(3)}(q)}{g''(q)}\right)^2$$

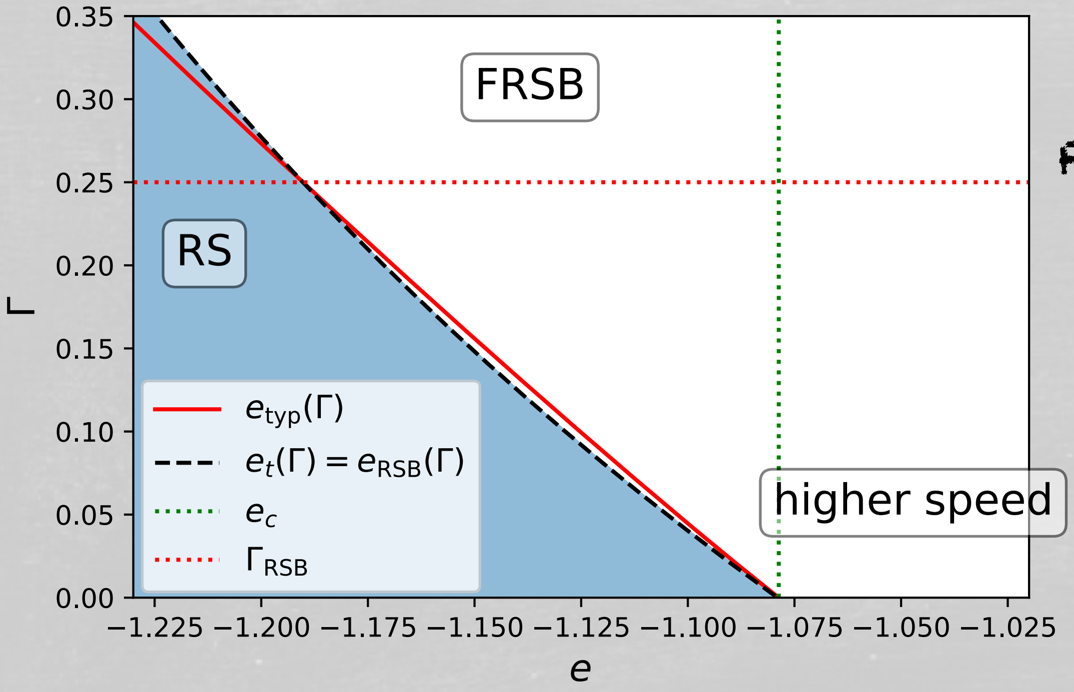
If $\forall q \in [0,1]$ one has $\mathcal{S}[g'(q)] > 0$ the model is FRSB

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF

$$\mathscr{L}(e) = -\min_s \left[se + \phi(s)\right] \leq \underbrace{-\Sigma_{\min}(e) = -\frac{1}{N}\ln \overline{\mathcal{N}_{\min}(e)}}_{\text{Bound from complexity}}$$



For any $\Gamma \geq 0$, there is both an RS and FRSB branch

One eigenvalues of $A_{(ab),(cd)} = \dfrac{\delta^2 \Phi_{n=s/\beta}(Q)}{\delta Q_{ab} \delta Q_{cd}}$

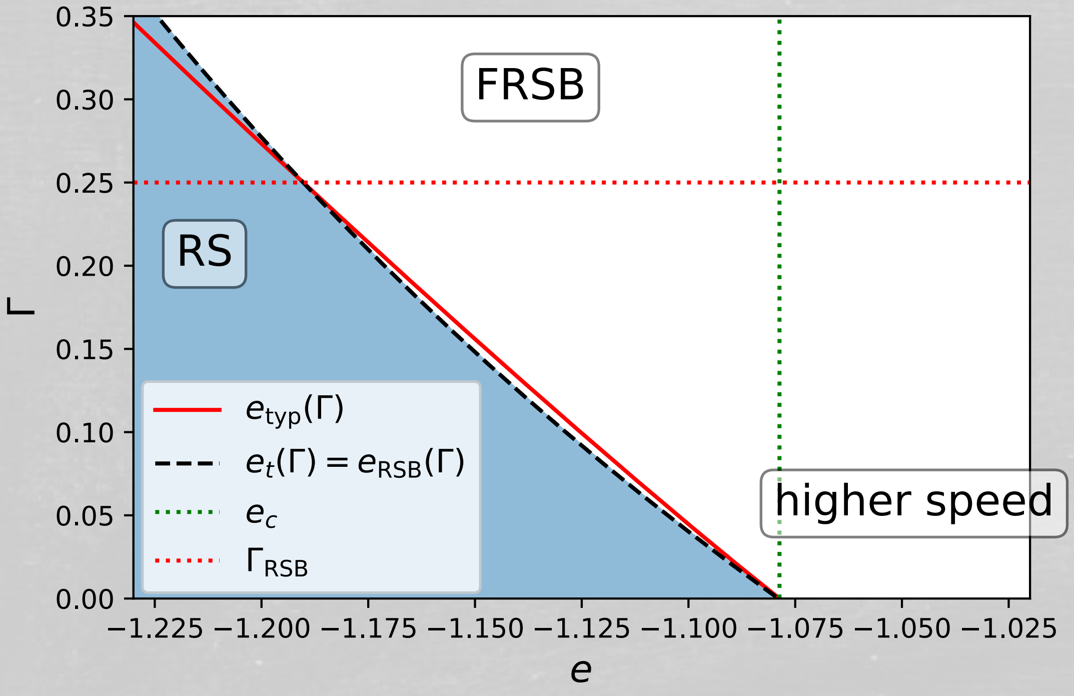
becomes positive for $e > e_{\mathrm{RSB}}$

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF

$$\mathcal{L}(e) = -\min_s \left[se + \phi(s)\right] \leq \underbrace{-\Sigma_{\min}(e) = -\frac{1}{N}\ln \overline{\mathcal{N}_{\min}(e)}}_{\text{Bound from complexity}}$$



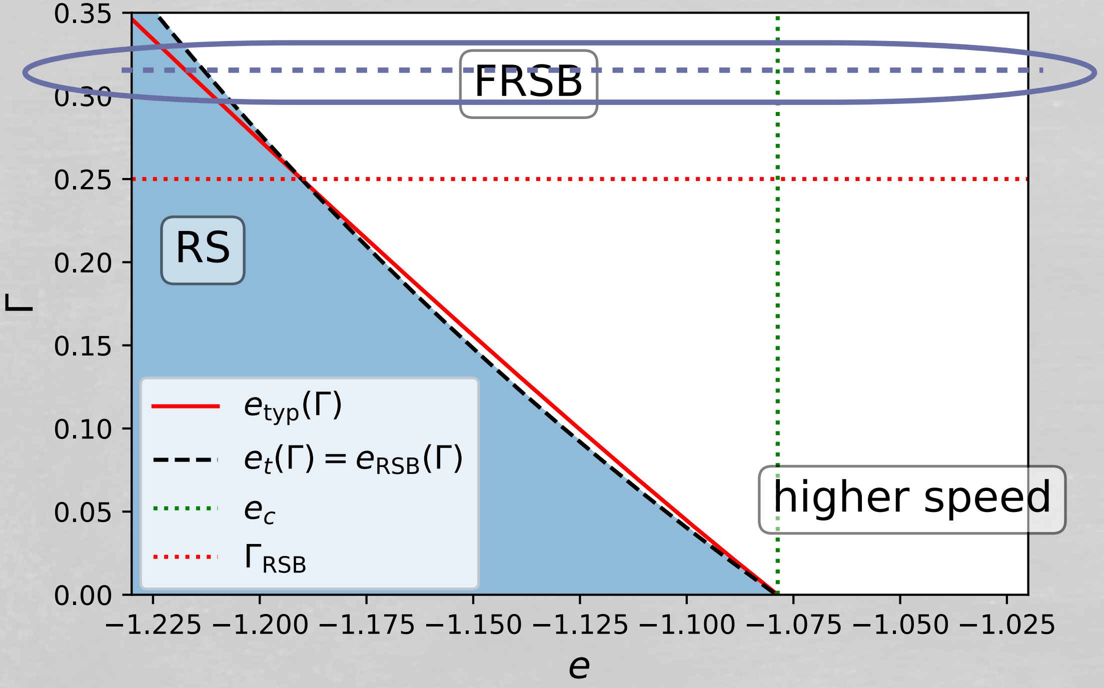
In the RS phase, the LDF saturates the bounds

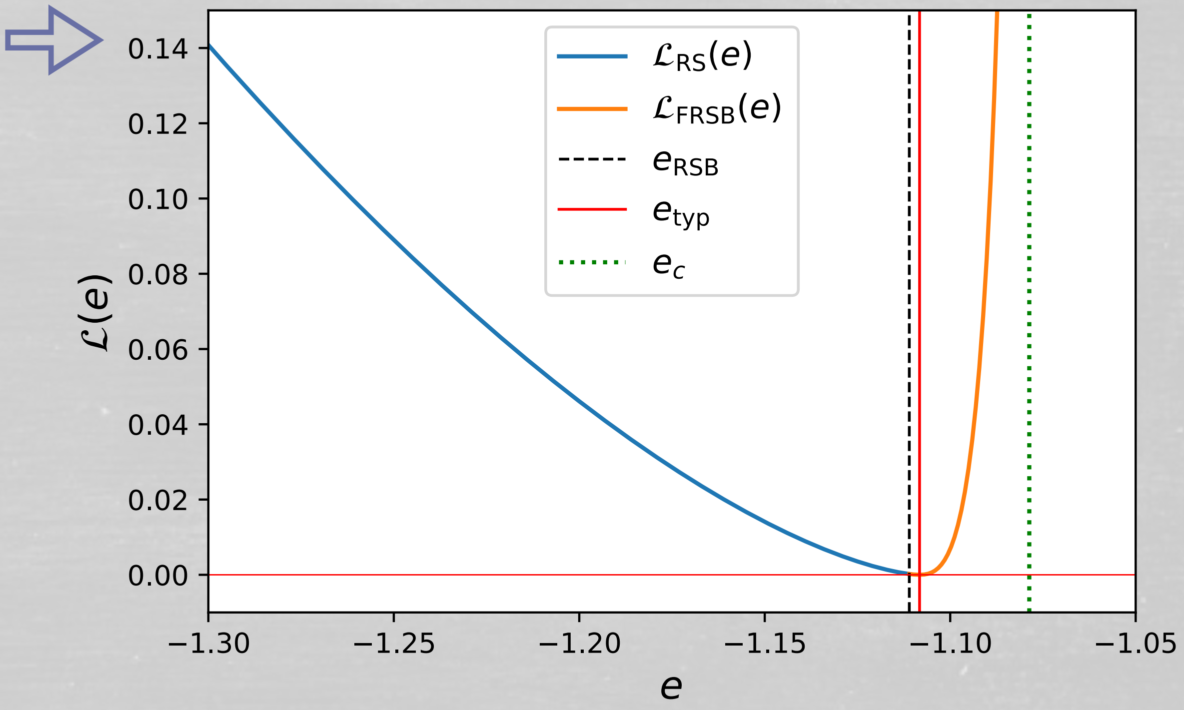
$$\mathcal{L}_{RS}(e) = -\lim_{N\to\infty}\frac{1}{N}\ln P_N(e) = -\Sigma_{\min}(e) = -\frac{1}{N}\ln \overline{\mathcal{N}_{\min}(e)}$$

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF $\mathscr{L}(e) = -\min_s \left[se + \phi(s) \right]$

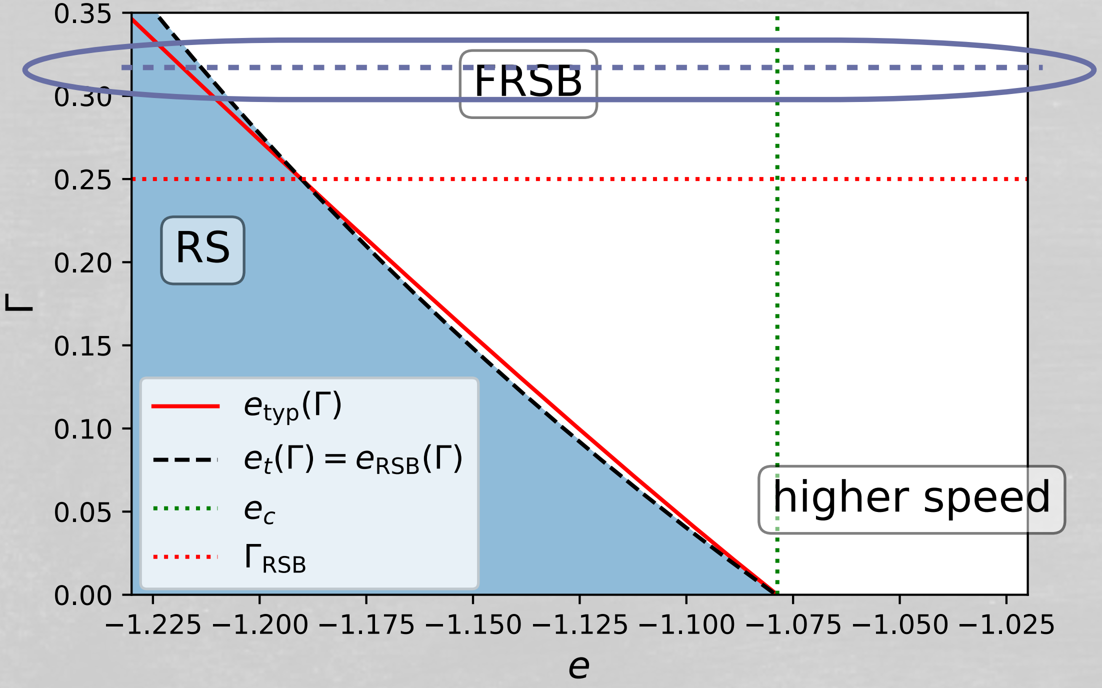


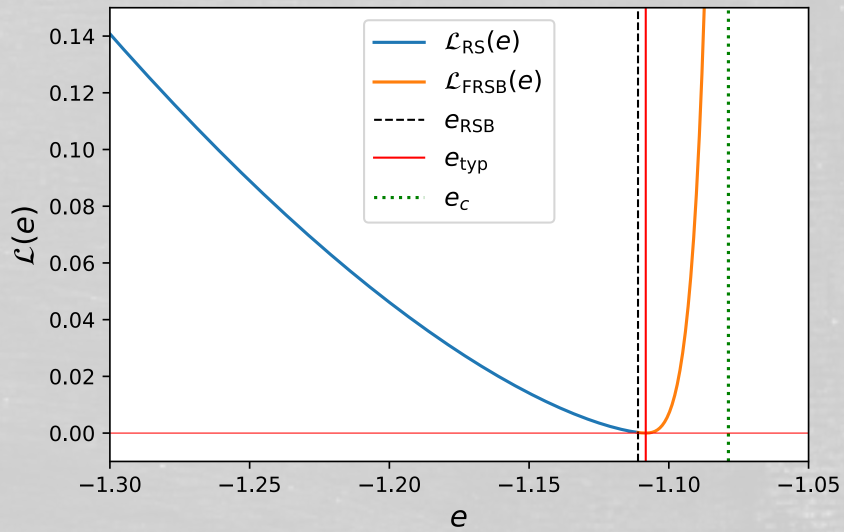


(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF





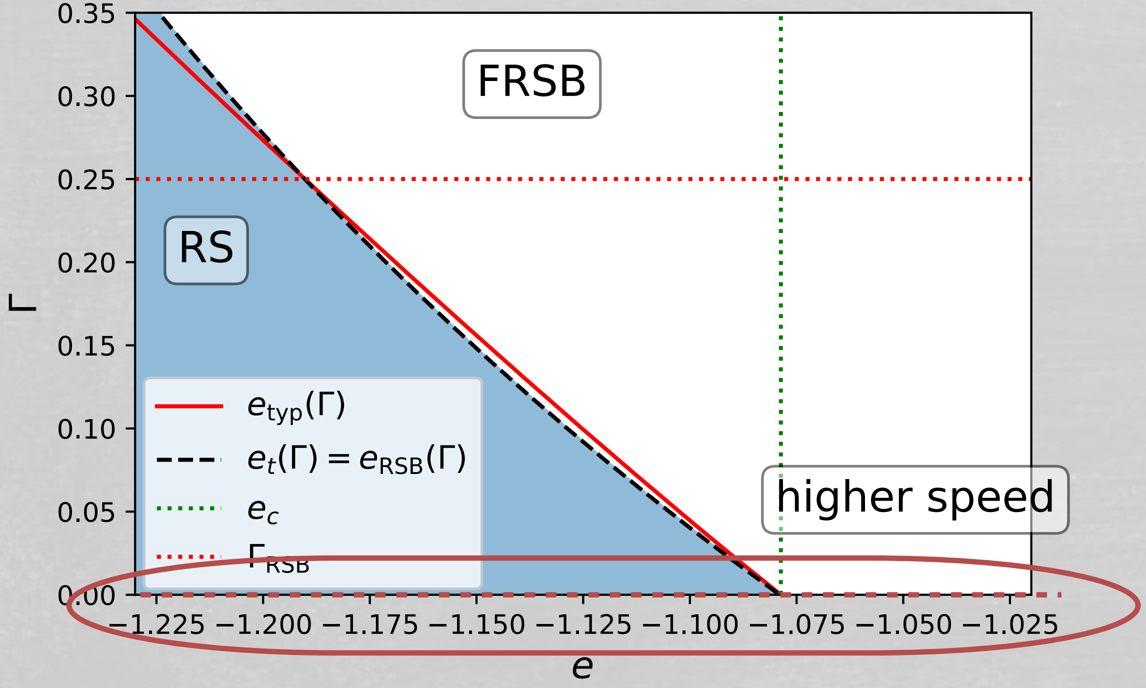
For $\Gamma > 0$, the LDF is quadratic around e_{typ} matching the Gaussian distribution

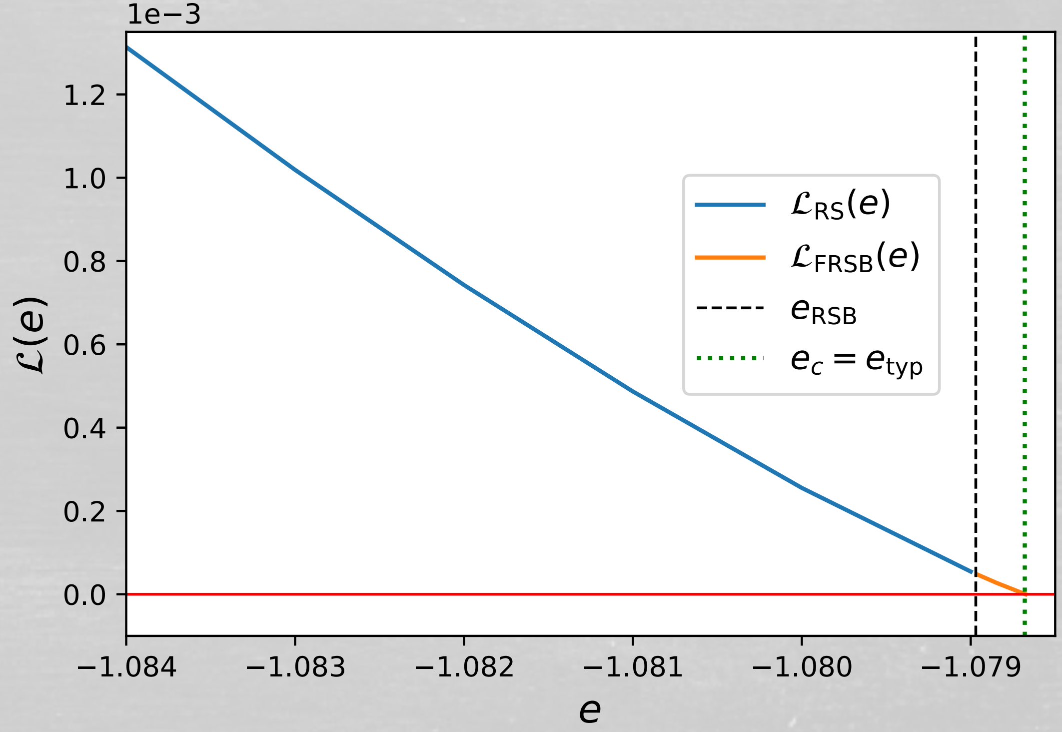
$$\mathcal{L}(e) \approx \frac{\left(e - e_{typ}\right)^2}{2\mathcal{V}_{min}}, \; e \to e_{typ}$$

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF

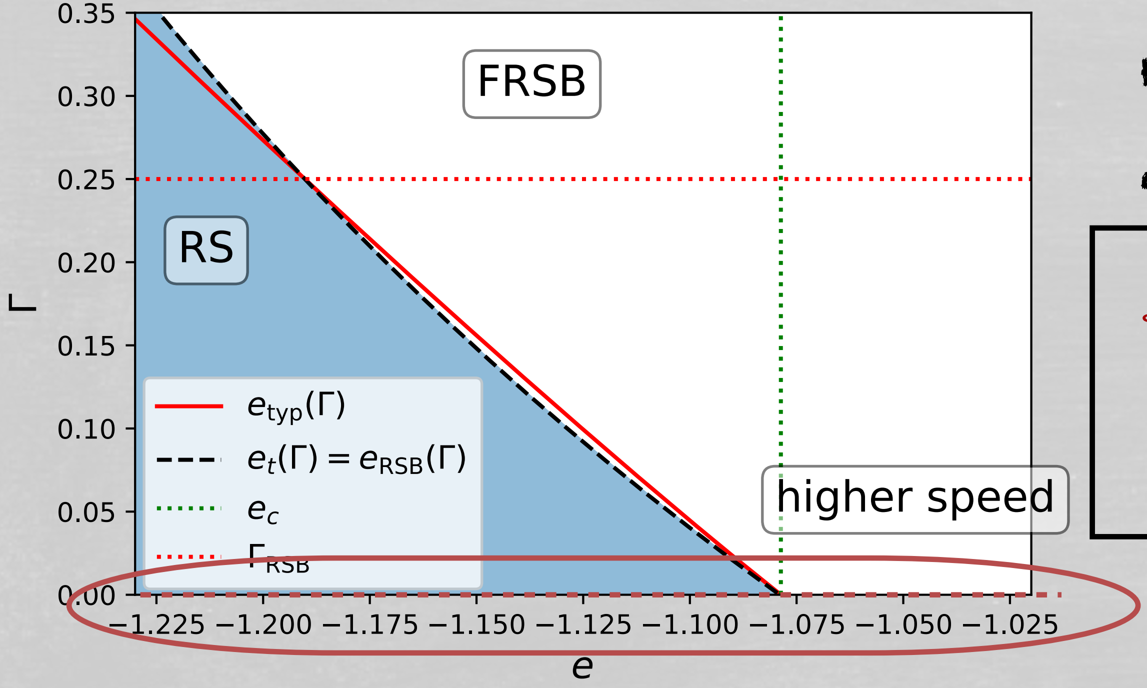




(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF $\mathscr{L}(e) = -\min_s \left[se + \phi(s) \right]$



For $\Gamma = 0$, the behaviour of the LDF depends on the covariance $g(q) \approx g_2 q^2 + g_r q^r \,, \; q \to 0$

$$\mathscr{L}(e) = \xi_r |e - e_c|^{\eta_r} \,, \;\; e \to e_{\mathrm{typ}}(\Gamma = 0) = e_c$$

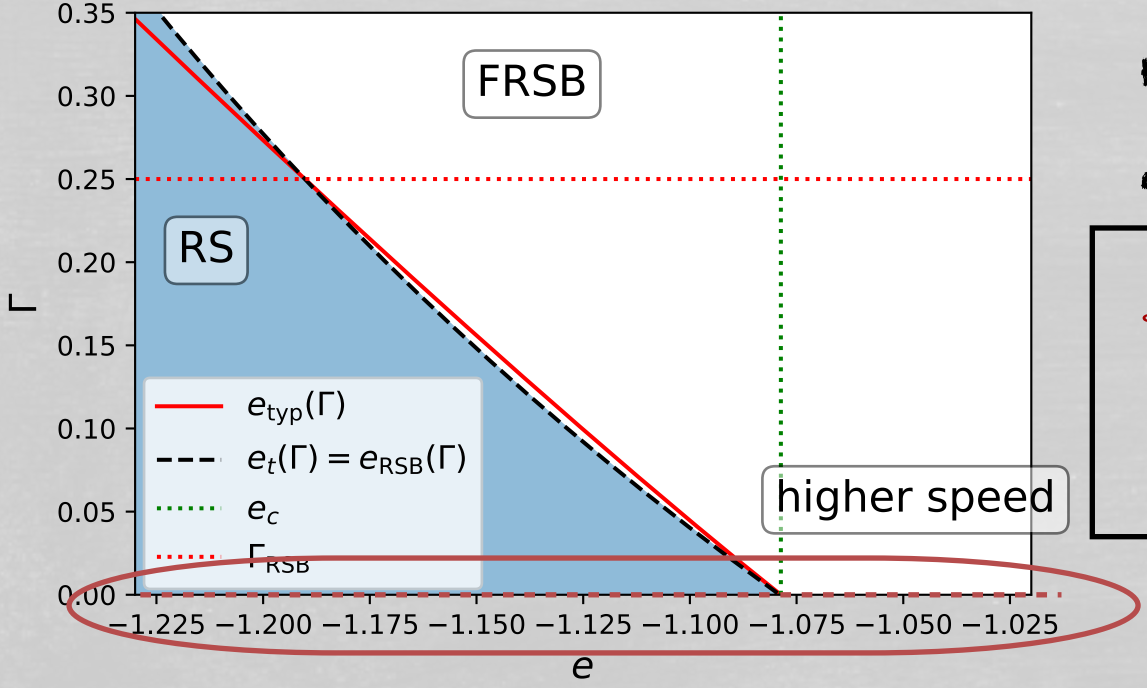
$$1 \leq \eta_r = \frac{3(r-2)}{2r-3} \leq \frac{3}{2}$$

By matching, new family of PDF for extreme values statistics with universal tails!

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF $\mathscr{L}(e) = -\min_s \left[se + \phi(s)\right]$



For $\Gamma = 0$, the behaviour of the LDF depends on the covariance $g(q) \approx g_2 q^2 + g_r q^r\,,\ q \to 0$

$$\mathscr{L}(e) = \xi_r |e - e_c|^{\eta_r}\,,\ e \to e_{typ}(\Gamma = 0) = e_c$$

$$1 \leq \eta_r = \frac{3(r-2)}{2r-3} \leq \frac{3}{2}$$

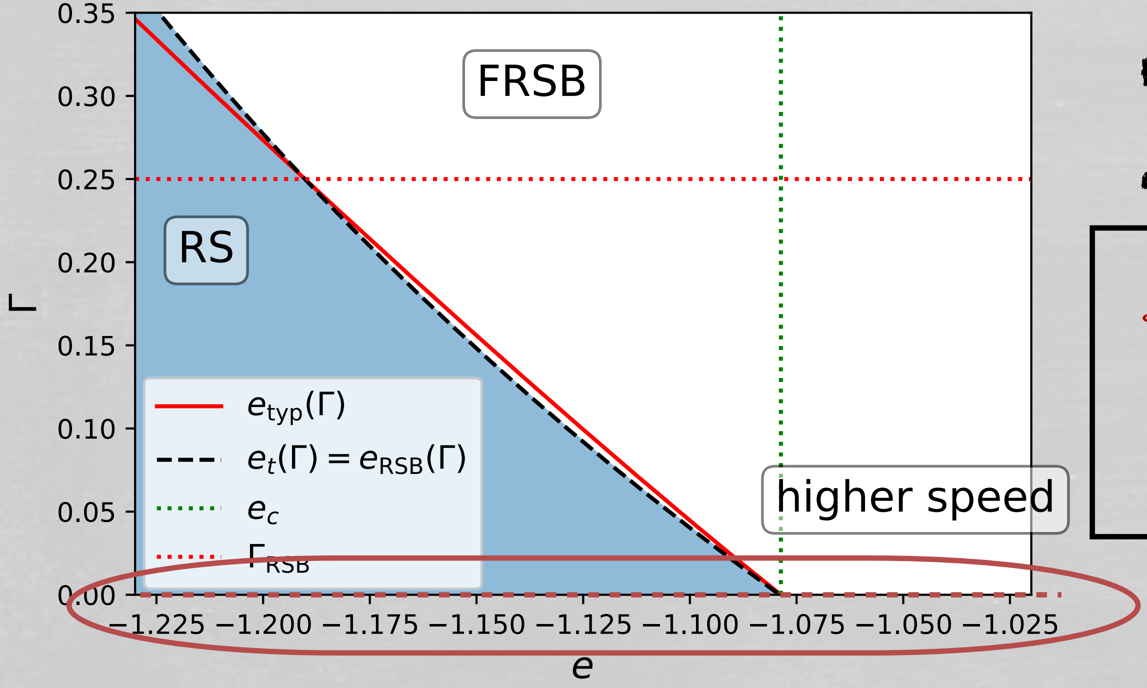
For $r = 3$, exponential tail $\eta_3 = 1$

For $r \to \infty$, TW tail $\eta_\infty = \frac{3}{2}$

(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (FRSB)

For a model with FRSB phase, one obtains for the LDF $\mathscr{L}(e) = -\min_s \left[se + \phi(s)\right]$



(LACT, Fyodorov & Le Doussal '23)

For $\Gamma = 0$, the behaviour of the LDF depends on the covariance $g(q) \approx g_2 q^2 + g_r q^r\,,\ q \to 0$

$$\mathscr{L}(e) = \xi_r |e - e_c|^{\eta_r}\,,\ \ e \to e_{\mathrm{typ}}(\Gamma = 0) = e_c$$

$$1 \leq \eta_r = \frac{3(r-2)}{2r-3} \leq \frac{3}{2}$$

For $r = 4$, same exponent observed for the Sherrigton-Kirkpatrick model

$\eta_4 = 6/5$

(Parisi & Rizzo '08)

Contents

Conclusion

We have studied systematically the atypical fluctuations of the ground-state energy of spherical spin-glasses

- Similar but more complex optimisation problem than for the typical GSE
- The large deviation of speed N is characterised by a rich phase diagram
- Replica-symmetry breaking may occur even if the typical GSE is replica-symmetric
- The study indicates the existence of new non-trivial universal distribution for the extreme value statistics of random landscapes
- The RS ansatz coincides with the opposite of the annealed complexity

To go further

Many directions to consider

- Fixed magnetic field
- Large deviation function with higher speed $s_N \gg N$ (at least at zero magnetic field)

SK: Parisi & Rizzo '10

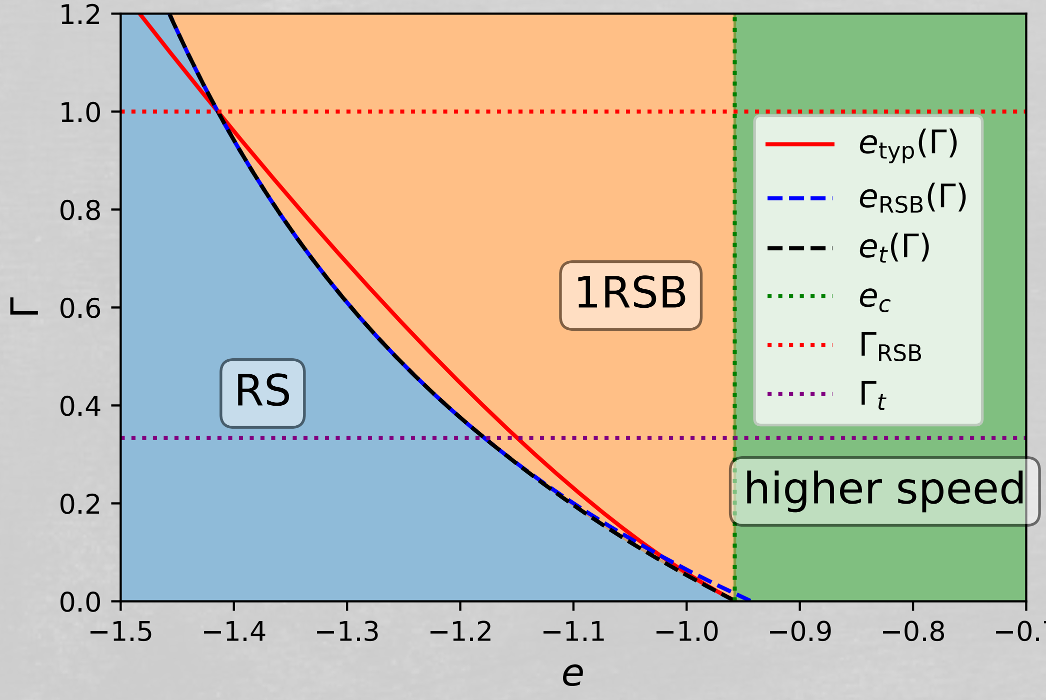
- Non mean-field / sparse models

SK: Parisi & Rizzo '09

- Study in more detail connection to complexity

Phase diagram for LDF (1RSB)

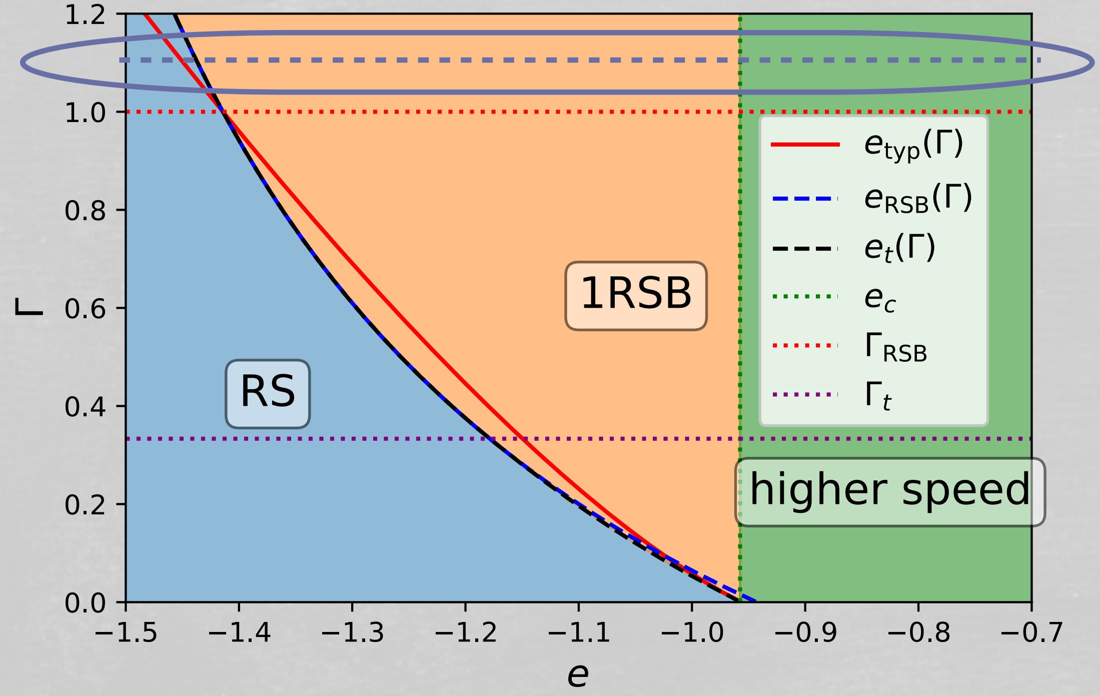
For a model with 1RSB phase, one obtains for the LDF

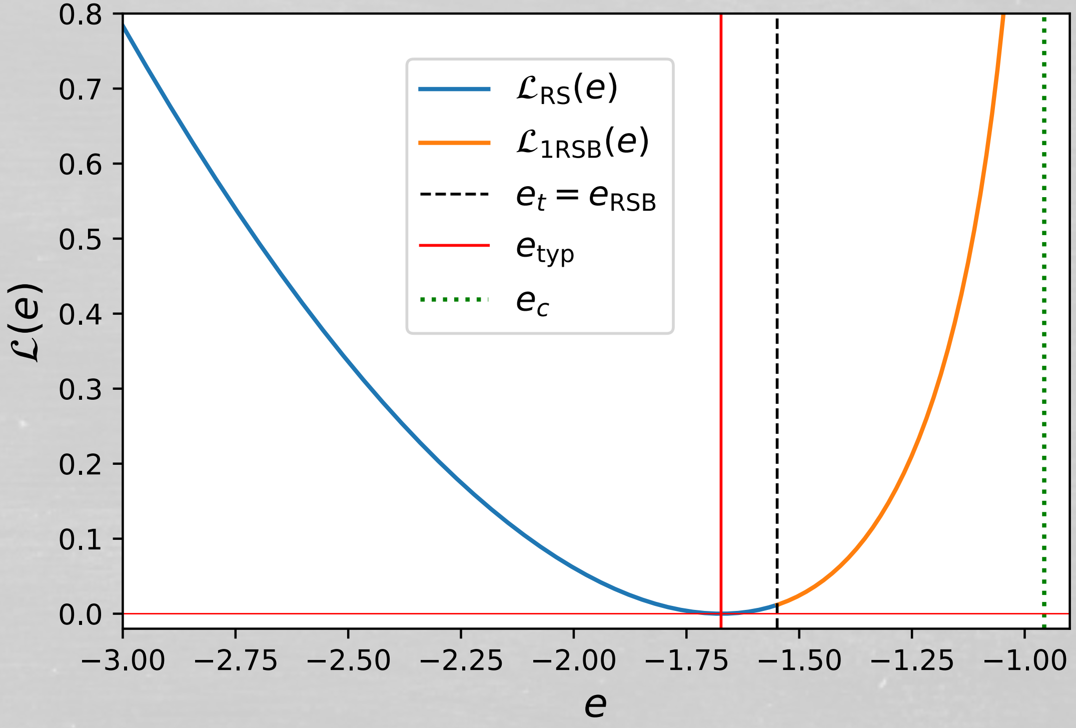


(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (1RSB)

For a model with 1RSB phase, one obtains for the LDF

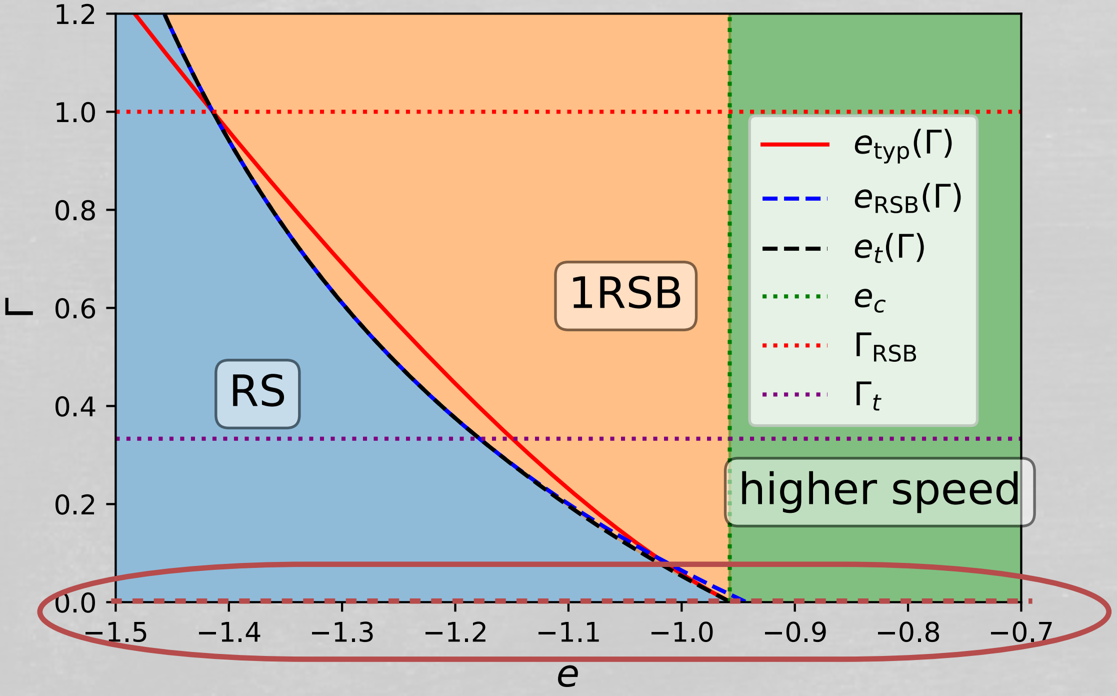


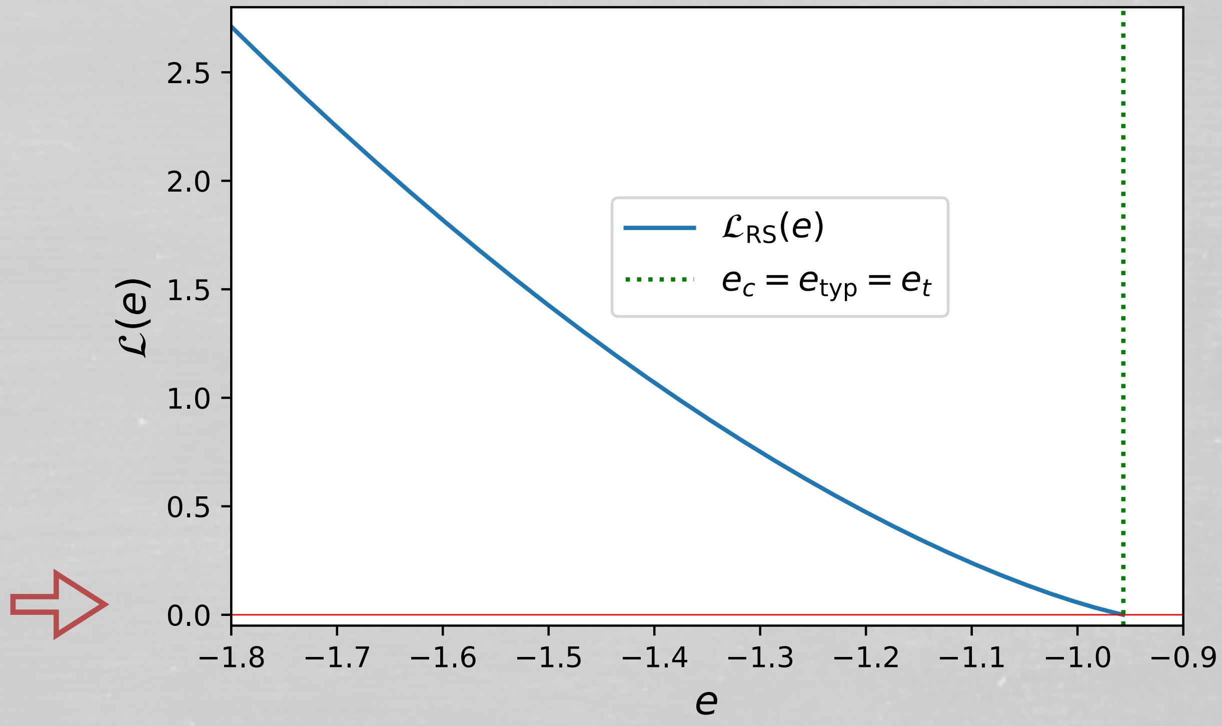


(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (1RSB)

For a model with 1RSB phase, one obtains for the LDF

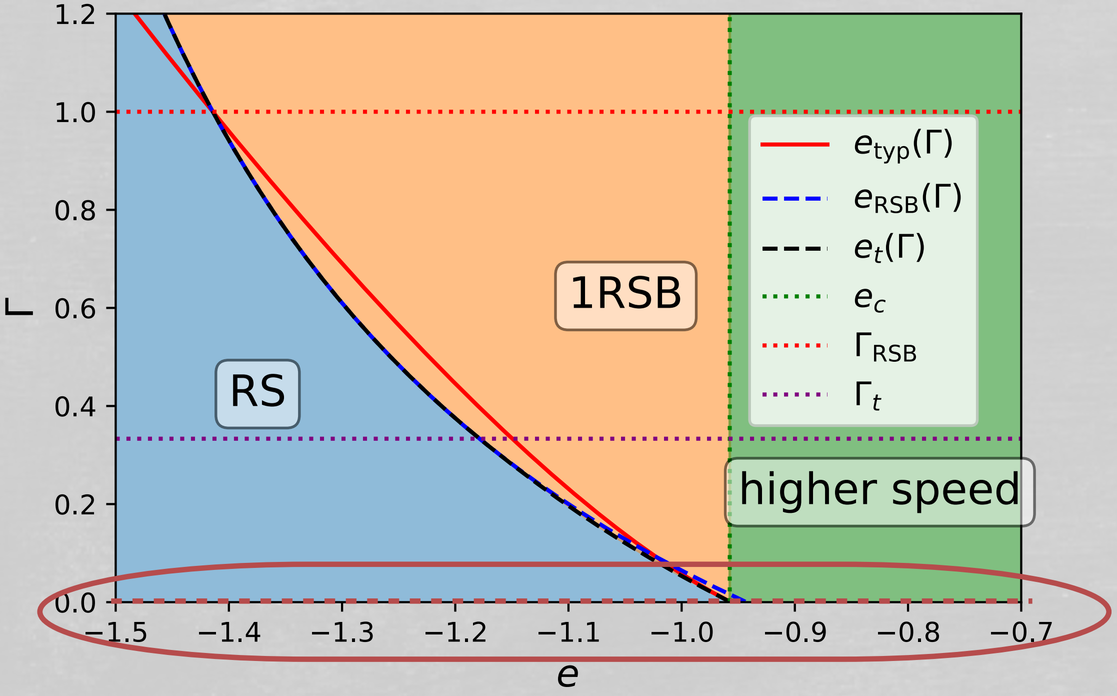




(LACT, Fyodorov & Le Doussal '23)

Phase diagram for LDF (1RSB)

For a model with 1RSB phase, one obtains for the LDF



In the RS phase, the LDF saturates the bounds

$$\mathscr{L}_{RS}(e) = -\lim_{N\to\infty}\frac{1}{N}\ln P_N(e) = -\Sigma_{min}(e) = -\frac{1}{N}\ln\overline{\mathscr{N}_{min}(e)}$$

$$\mathscr{L}_{RS}(e_c) = -\Sigma_{min}(e_c) = 0$$

For $\Gamma = 0$, the behaviour of the LDF is linear

$$\mathscr{L}(e) = -\Sigma'_{min}(e_c)\,|e - e_c|\,,\quad e \to e_{typ}(\Gamma = 0) = e_c$$

(LACT, Fyodorov & Le Doussal '23)