The giant component, articulation points and bredges in configuration model networks

Ofer Biham

The Racah Institute of Physics

The Hebrew University of Jerusalem

In collaboration with
Eytan Katzav, Reimer Kühn, Haggai Bonneau and Ido Tishby

Supported by the Israel Science Foundation grant 1682/18

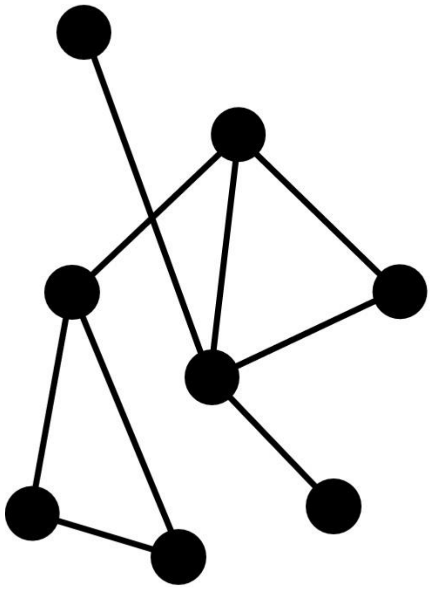
Outline

- Random networks: the configuration model
- The microstructure of the giant component
- Articulation points (APs) and bredges
- Statistical analysis and correlations

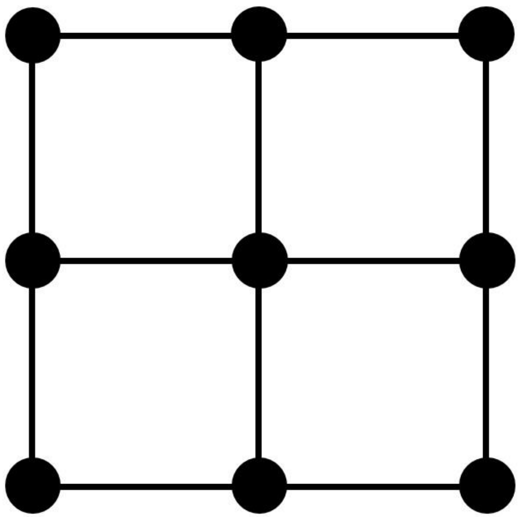
Random Networks

Random networks (graphs) consist nodes that are connected to each other by edges according to some stochastic rule.

Random network

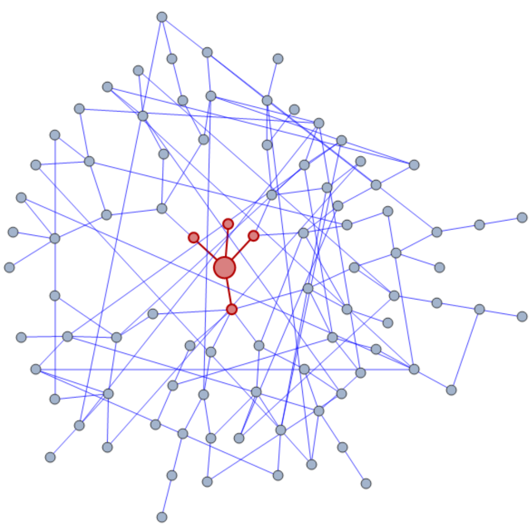


Lattice

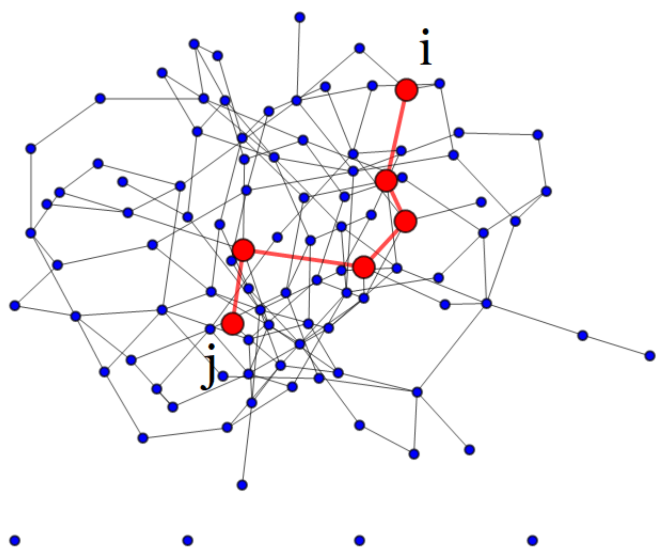


Random Networks

Microstructure:
degree distribution



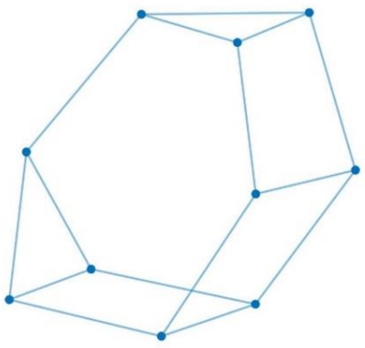
Large-scale structure:
the distribution of shortest path lengths (DSPL)



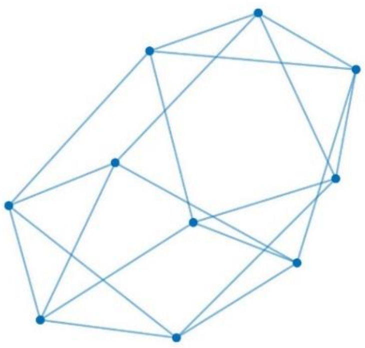
Random Networks/Graphs

In random regular graphs (RRGs) all the nodes are of the same degree $c \geq 3$.

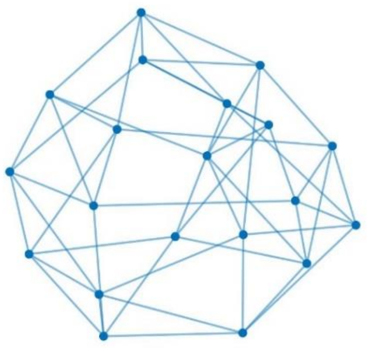
The network consists of a single connected component of size N.



$c = 3$
$N = 10$



$c = 4$
$N = 10$

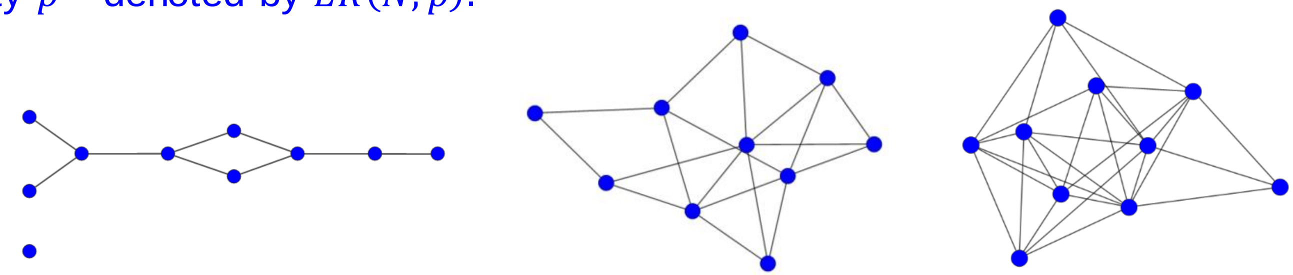


$c = 5$
$N = 20$

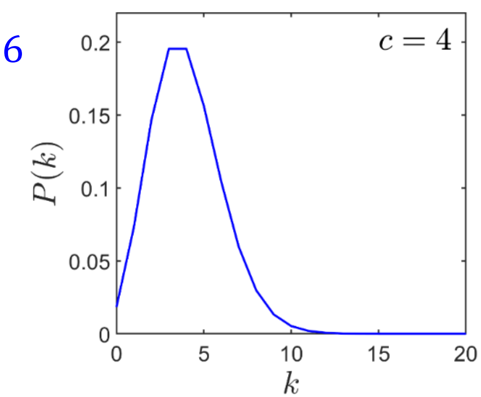
Random Networks

Erdős–Rényi (ER) graphs: N nodes, where each pair of nodes is connected independently with probability p – denoted by $ER(N,p)$.

$N = 10$



$p = 0.2$ $\qquad$ $p = 0.4$ $\qquad$ $p = 0.6$

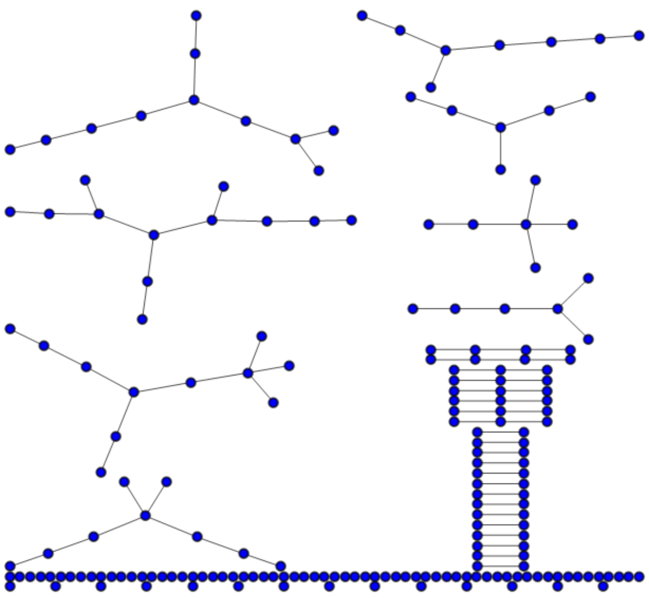


- Binomial degree distribution $P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$
- Binomial -> Poisson degree distribution: $P(k) = \frac{e^{-c}c^k}{k!} \equiv \pi(k)$, k = 0,1,2,3,…, where $c = \langle K \rangle = (N-1)p$ is the mean degree.
- No degree-degree correlations or any other correlations.

Erdős-Rényi (ER) networks

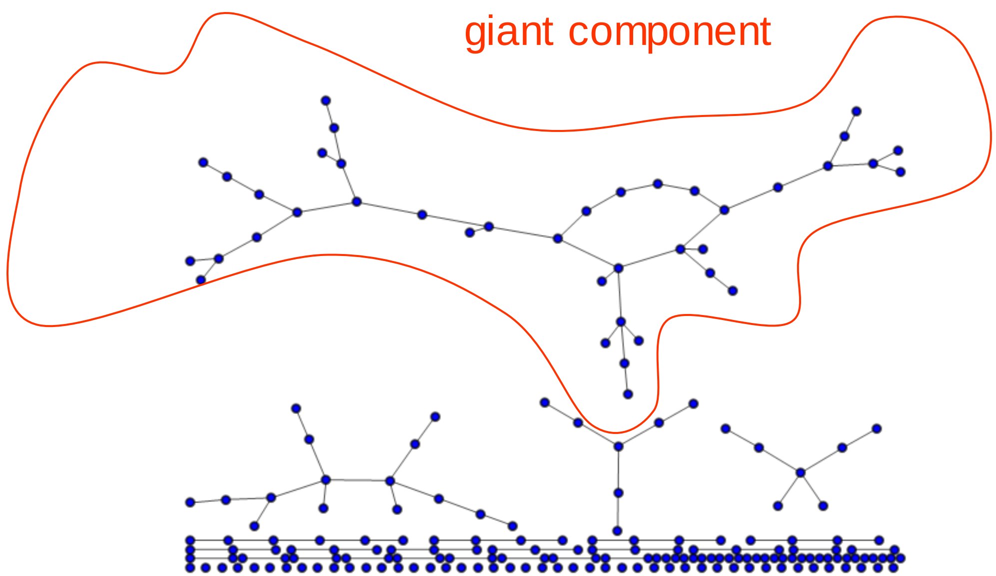
An ER network consists of N nodes, where each pair of nodes is connected with probability p. The mean degree is $c = (N-1)p$.

subcritical regime: $c < 1$



N = 200, c = 0.9

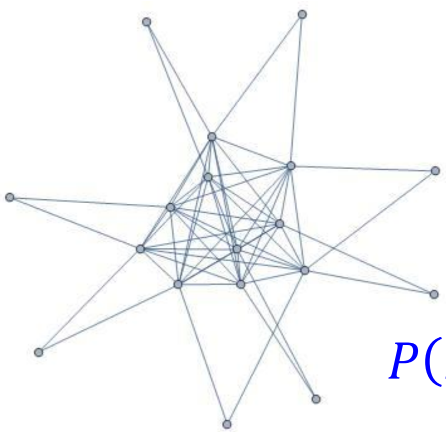
supercritical regime: $c > 1$



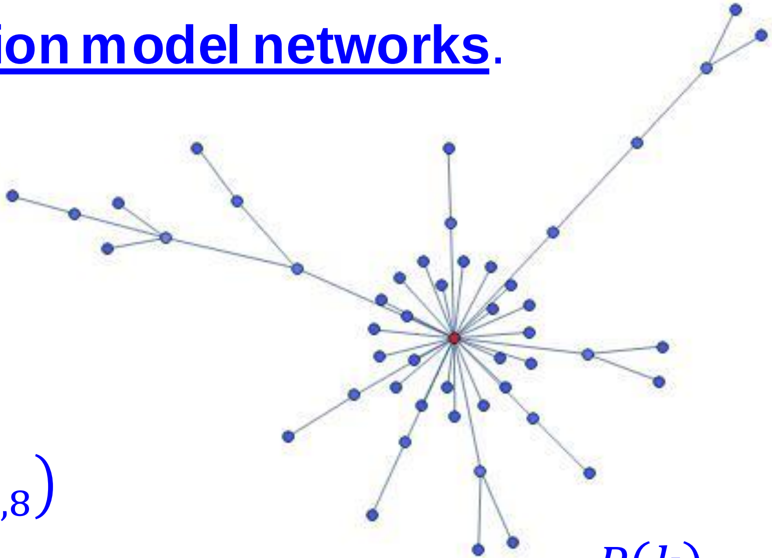
N = 200, c = 1.1

The Configuration Model

One can impose any degree sequence, obtained from a certain **degree distribution $P(k)$**. This can be an exponential, Gaussian, Poisson or a power-law distribution. In these networks there are no degree-degree correlations. They form a *maximum entropy ensemble*, and are referred to as **Configuration model networks**.



$$P(k) = \frac{1}{2}\left(\delta_{k,2} + \delta_{k,8}\right)$$



$$P(k) = A\, k^{-\gamma}$$

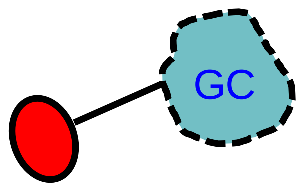
The giant component

ER networks: percolation transition at $c = 1$. It has two order parameters:

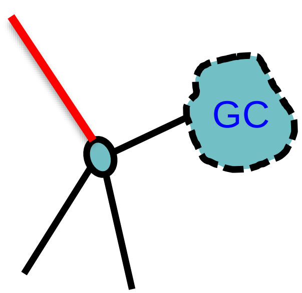
g : probability that a randomly selected node resides on the giant component

$\tilde{g}$: probability that a node selected via a random edge resides on the giant component

g



$\tilde{g}$



Self-consistent equations:

$$1 - \tilde{g} = G_1(1 - \tilde{g})$$

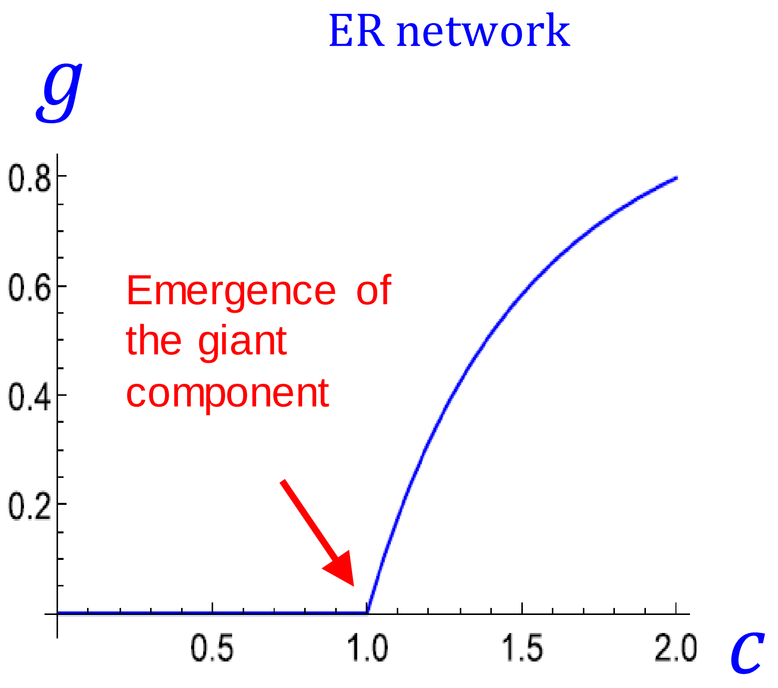
$$1 - g = G_0(1 - \tilde{g})$$

Generating functions:

$$G_0(x) = \sum_{k=0}^{\infty} x^k P(k)$$

$$G_1(x) = \sum_{k=1}^{\infty} x^{k-1} \frac{k}{c} P(k)$$

The Giant Component of ER Networks

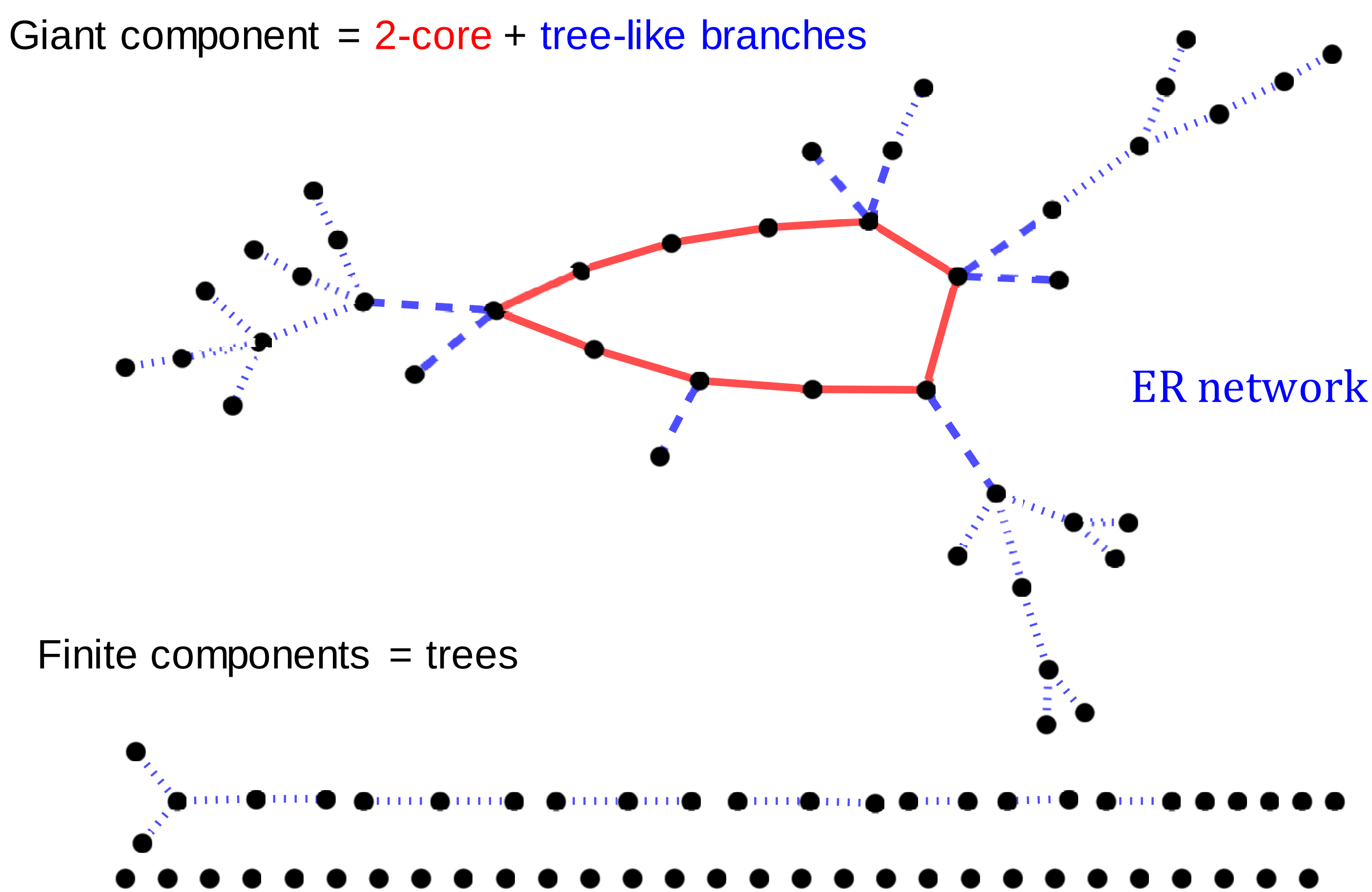


$$P(k|\mathrm{GC}) = \frac{1-(1-\tilde{g})^k}{g} P(k)$$

$$g = 1 + \frac{W(-ce^{-c})}{c}$$

I. Tishby, O. Biham, E. Katzav and R. Kühn, Revealing the microstructure of the giant component in random graph ensembles, *Phys. Rev.* E **97**, 042318 (2018)

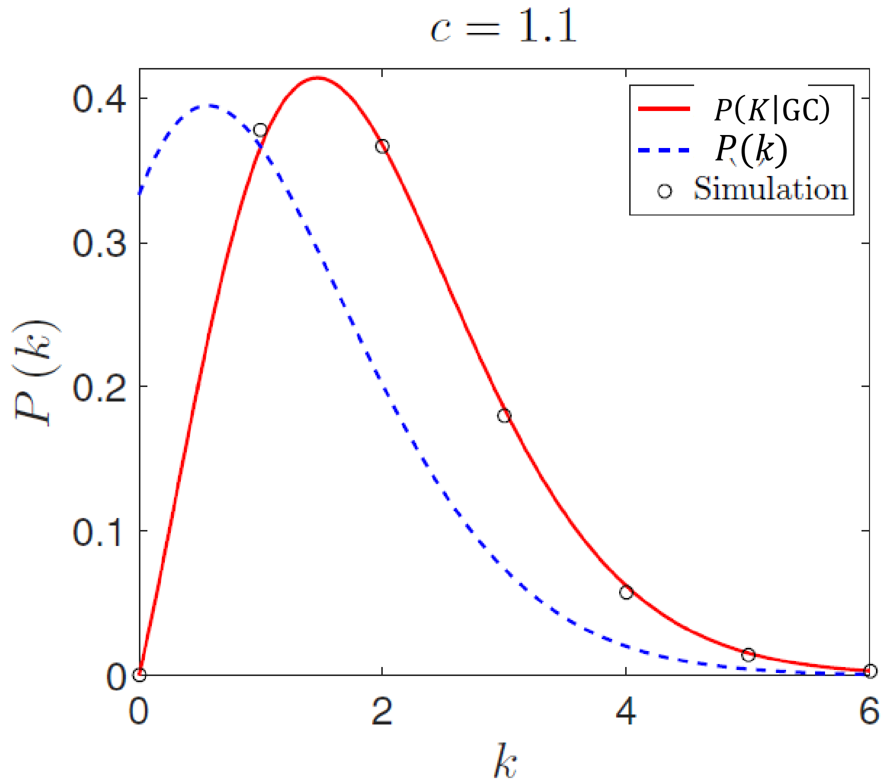
The structure of the giant component

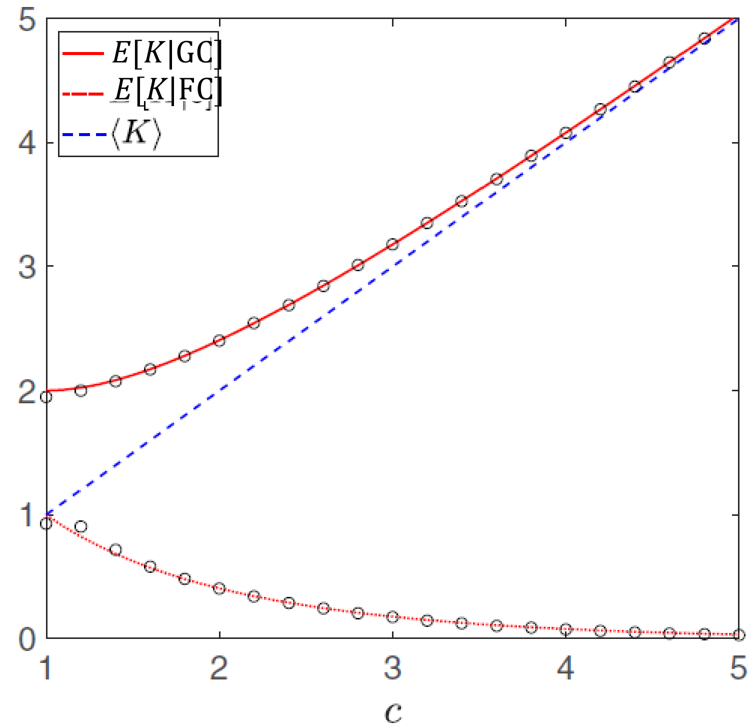


The Degree Distribution on the GC

For example for ER:

$$P(K|\mathrm{GC}) = \frac{1}{g}\left[\frac{e^{-c}c^k}{k!} - (1-g)\frac{e^{-c(1-g)}[c(1-g)]^k}{k!}\right] \qquad c_1 = (2-g)c$$

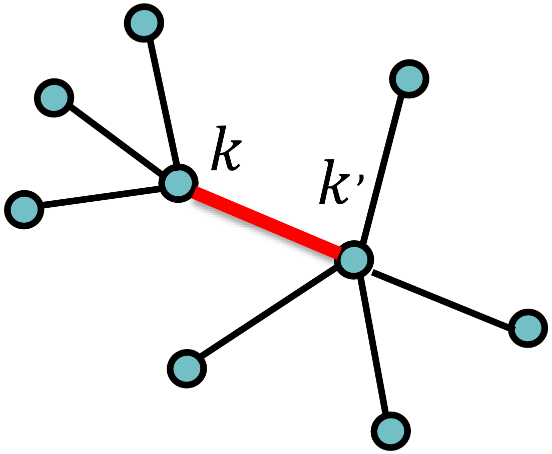


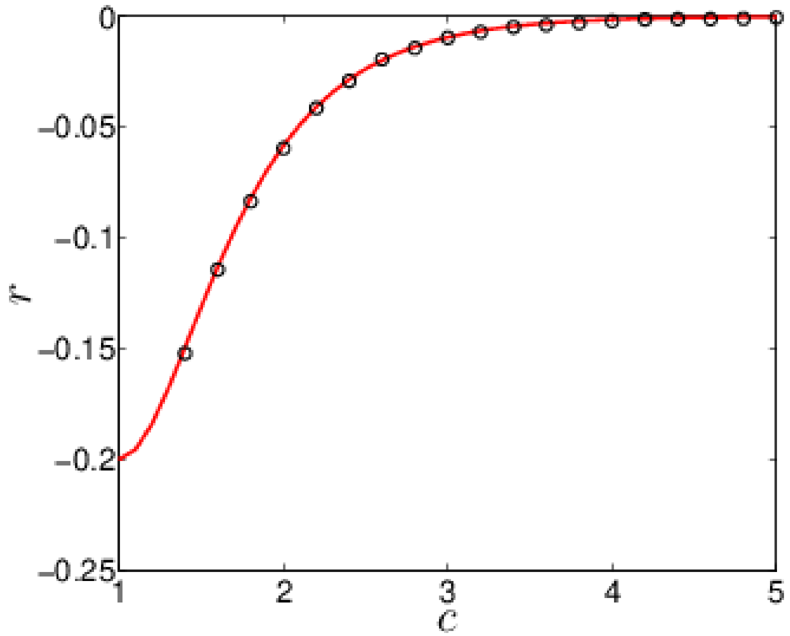


Degree-Degree Correlations on the GC

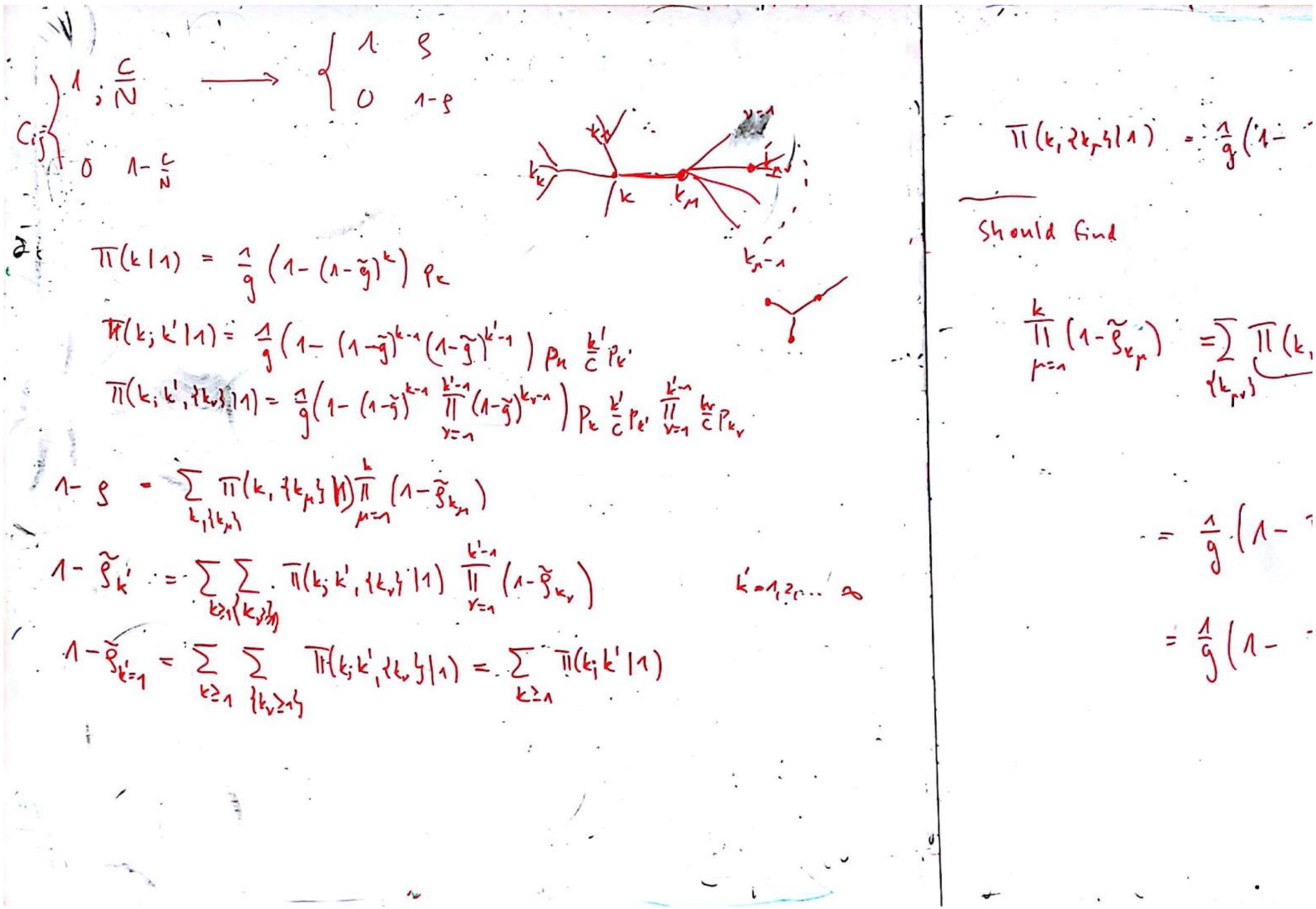
Assortativity Coefficient:

$$r = \frac{E[KK'|GC] - (E[K|GC])^2}{Var(K|GC)}$$





Reimer’s notes on the GC



Random Networks Consisting of a Single Connected Component

By inverting the relations between $P(k)$ and $P(k|1)$, we can infer $P(k)$ that will exhibit any desired $P(k|1)$. Thus, using the configuration model, a single component network of any desired degree distribution can be generated:

$$P(k) = \frac{g}{1-(1-\tilde{g})^k} P(k \mid 1)$$

I. Tishby, O. Biham, E. Katzav and R. Kühn, Generating random networks that consist of a single connected component with a given degree distribution, *Phys. Rev.* E **99**, 042308 (2019)

Single Component Network Models

Define desired size and degree distribution

$P(k \mid 1)$

$\langle N_1 \rangle$

$$P(k) = \frac{g}{1-(1-\tilde{g})^k} P(k \mid 1)$$

Solve self consistent equations (numerically or analytically)

$g \quad \tilde{g}$

$P(k), \langle N_1 \rangle / g$

Configuration Model Network

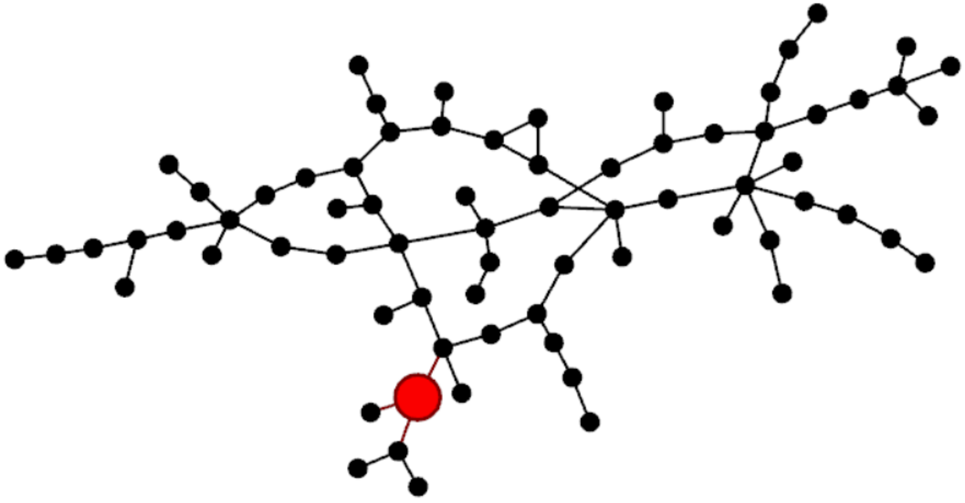
Generate Network

Single Component Network

Remove off-giant components

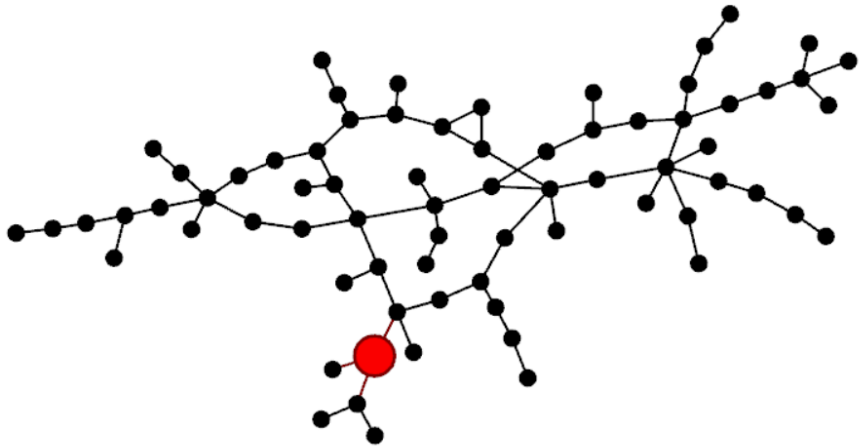
Articulation Points

Articulation Points (AP) are "weak points" in the network or "single points of failure". The deletion of an AP breaks up the network component on which it resides into two or more components.

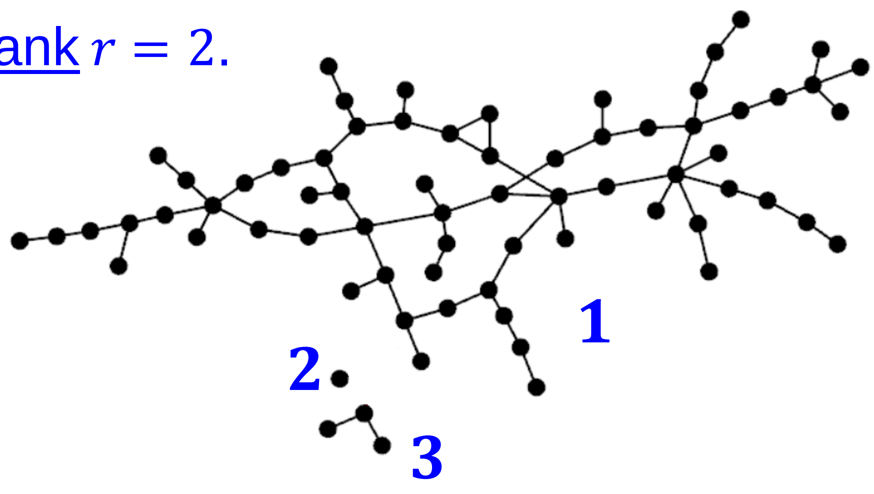


Articulation Points

Focusing on the giant component, consider the node marked in red, it is an Articulation Point:

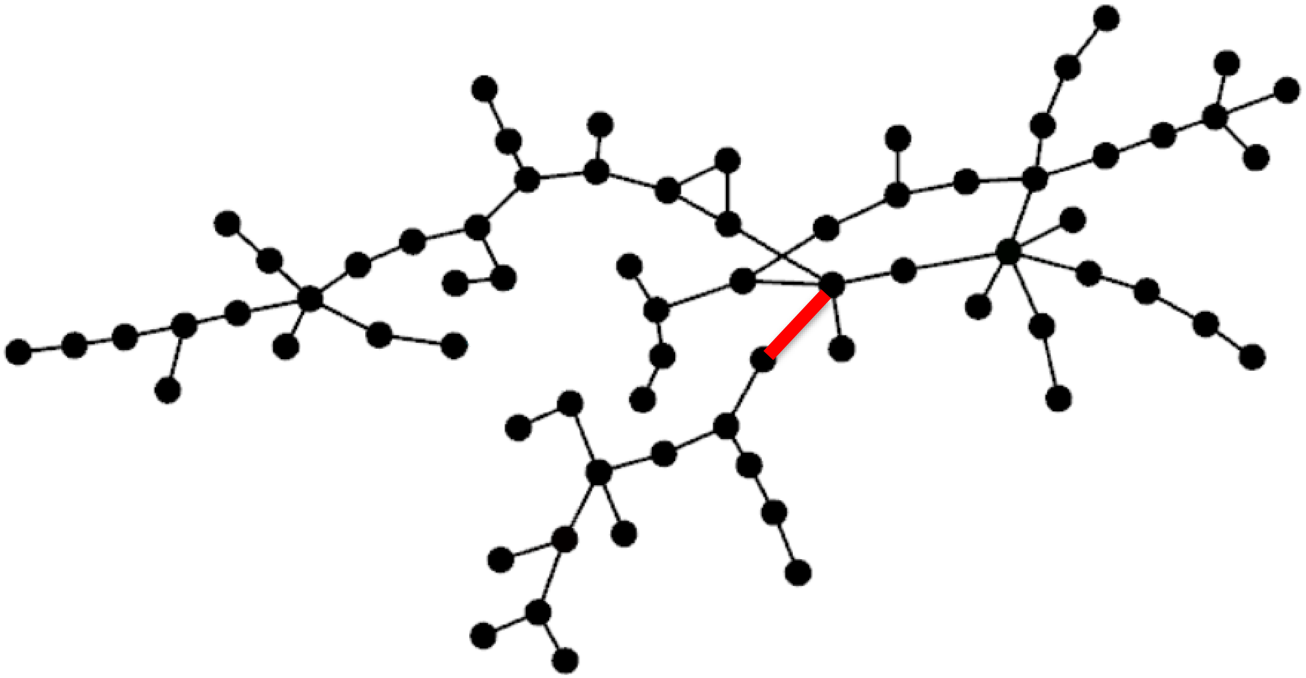


Removing it breaks up the giant component into three components, we say that it has an articulation rank $r = 2$.



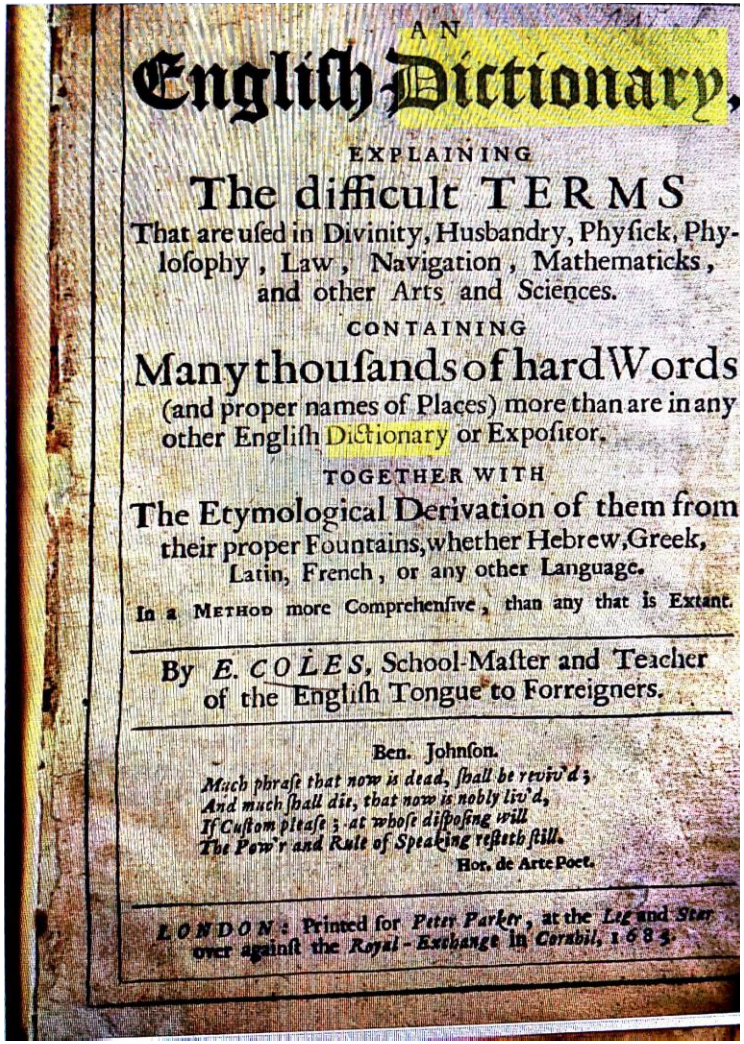
Bredges* (bridge-edges)

Bredges are “weak links” in the network. The deletion of a bredge breaks up the network component on which it resides into two components.



*English Dictionary, by E. Coles, School-Master and Teacher, printed at the Leg and Star over against the Royal-Exchange in Cornbil (1683).

Bredges


AN

English Dictionary,

EXPLAINING

The difficult TERMS

That are uſed in Divinity, Husbandry, Phyſick, Phy-loſophy, Law, Navigation, Mathematicks, and other Arts and Sciences.

CONTAINING

Many thouſands of hard Words

(and proper names of Places) more than are in any other Engliſh Dictionary or Expoſitor.

TOGETHER WITH

The Etymological Derivation of them from their proper Fountains, whether Hebrew, Greek, Latin, French, or any other Language.

In a METHOD more Comprehenſive, than any that is Extant.

By *E. COLES*, School-Maſter and Teacher of the Engliſh Tongue to Forreigners.

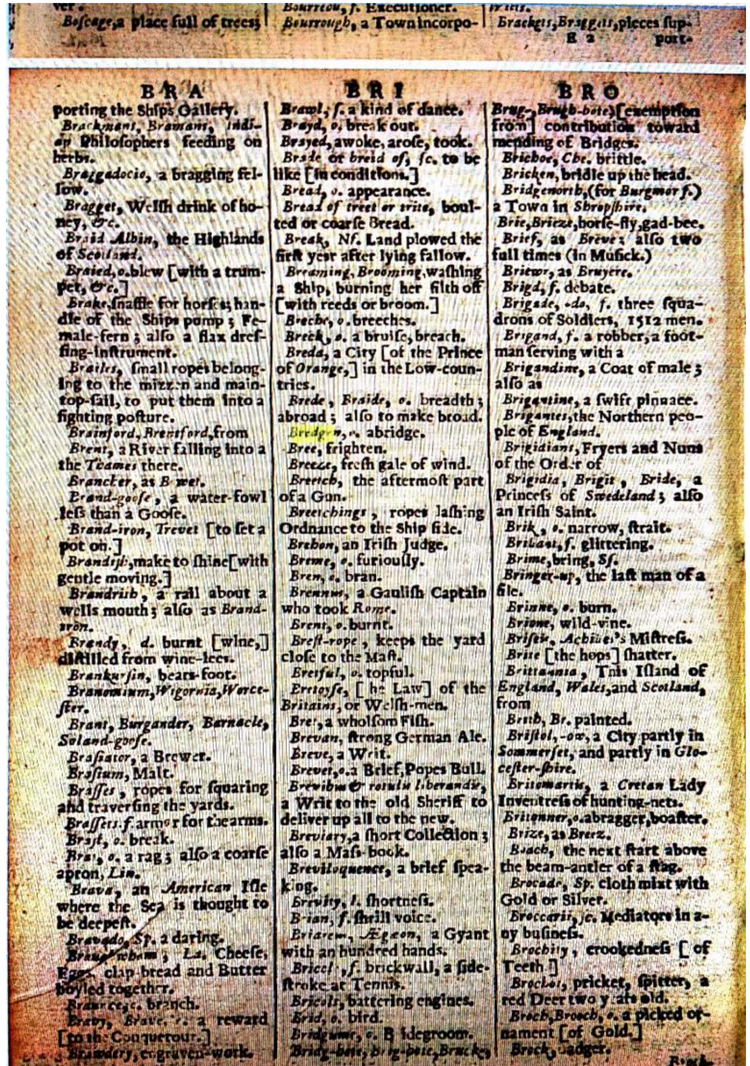
Ben. Johnſon.

Much phraſe that now is dead, ſhall be reviv'd;
And much ſhall die, that now is nobly liv'd,
If Cuſtom pleaſe; at whoſe diſpoſing will
The Pow'r and Rule of Speaking reſteth ſtill.

Hor. de Arte Poet.

LONDON: Printed for *Peter Parker*, at the *Leg and Star* over againſt the *Royal-Exchange* in *Cornhil*, 1685.

Bredges


ver.
Boſcage, a place full of trees;

Bourreau, ſ. Executioner.
Bourrough, a Town incorpo-

rates.
Brackets, Braggets, pieces ſup-
B 2 port-

B R A

porting the Ships Gallery.
Brachmans, Bramans, Indian Philoſophers feeding on herbs.
Braggadocio, a bragging fellow.
Bragget, Welſh drink of honey, &c.
Braid Albin, the Highlands of Scotland.
Braied, o. blew [with a trumpet, &c.]
Brake, ſnaffle for horſes; handle of the Ships pump; Female-fern; alſo a flax dreſſing-inſtrument.
Brailes, ſmall ropes belonging to the mizzen and main-top-ſail, to put them into a fighting poſture.
Brainford, Brentford, from
Brent, a River falling into a the Thames there.
Brancker, as B wet.
Brand-gooſe, a water-fowl leſs than a Gooſe.
Brand-iron, Trevet [to ſet a pot on.]
Brandiſh, make to ſhine [with gentle moving.]
Brandrith, a rail about a wells mouth; alſo as Brand-iron.
Brandy, d. burnt [wine,] diſtilled from wine-lees.
Brankurſin, bears-foot.
Branonium, Wigornia, Worceſter.
Brant, Burgander, Barnacle, Soland-gooſe.
Braſiator, a Brewer.
Braſium, Malt.
Braſſes, ropes for ſquaring and traverſing the yards.
Braſſets, f. armor for the arms.
Braſt, o. break.
Brat, o. a rag; alſo a coarſe apron, Lin.
Brava, an American Iſle where the Sea is thought to be deepeſt.
Bravado, Sp. a daring.
Braughwham, La. Cheeſe, Eggs, clip bread and Butter boyled together.
Braunce, c. branch.
Bravy, Brave, f. a reward [to the Conquerour.]
Braudery, engraven-work.

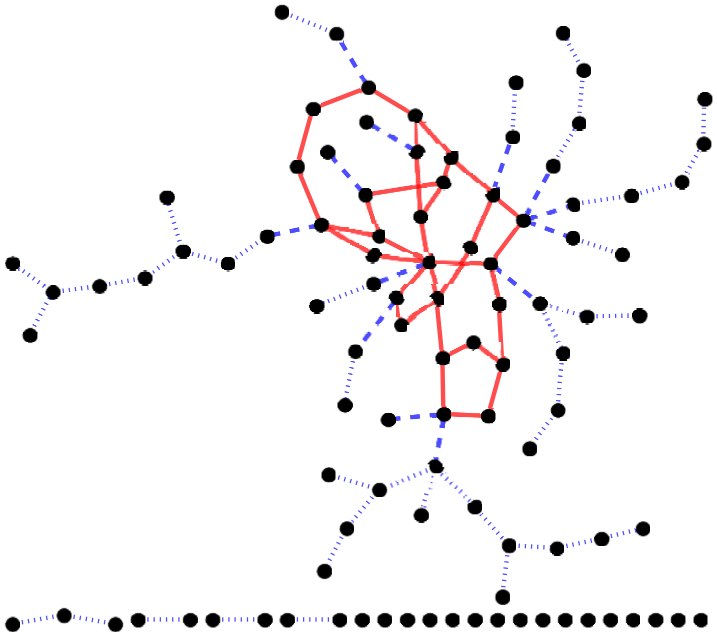
B R I

Brawl, f. a kind of dance.
Brayd, o. break out.
Brayed, awoke, aroſe, took.
Brede or breid of, ſc. to be like [in conditions.]
Bread, o. appearance.
Bread of treet or trite, boulted or coarſe Bread.
Break, Nf. Land plowed the firſt year after lying fallow.
Breaming, Brooming, waſhing a Ship, burning her filth off [with reeds or broom.]
Brechе, o. breeches.
Breck, o. a bruiſe, breach.
Breda, a City [of the Prince of Orange,] in the Low-countries.
Brede, Braide, o. breadth; abroad; alſo to make broad.
Bredgen, o. abridge.
Bree, frighten.
Breeze, freſh gale of wind.
Breech, the aftermoſt part of a Gun.
Breechings, ropes laſhing Ordnance to the Ship ſide.
Brehon, an Iriſh Judge.
Breme, o. furiouſly.
Bren, o. bran.
Brennus, a Gauliſh Captain who took Rome.
Brent, o. burnt.
Breſt-rope, keeps the yard cloſe to the Maſt.
Bretful, o. topful.
Bretoyſe, [the Law] of the Britains, or Welſh-men.
Bret, a wholſom Fiſh.
Brevan, ſtrong German Ale.
Breve, a Writ.
Brevet, o. a Brief, Popes Bull.
Brevibus & rotulis liberandis, a Writ to the old Sheriff to deliver up all to the new.
Breviary, a ſhort Collection; alſo a Maſs-book.
Breviloquence, a brief ſpeaking.
Brevity, l. ſhortneſs.
Brian, f. ſhrill voice.
Briareus, Ægeon, a Gyant with an hundred hands.
Bricol, f. brickwall, a ſide-ſtroke at Tennis.
Bricols, battering engines.
Brid, o. bird.
Bridgume, o. Bridegroom.
Brigg-bote, brig-bote, Brucke

B R O

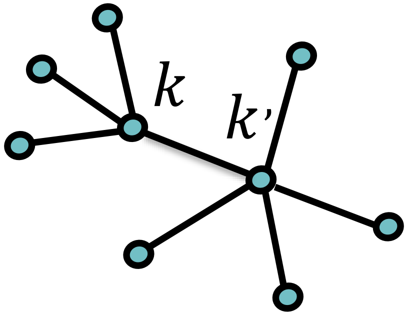
Brug-, Brugh-bote) [exemption from] contribution toward mending of Bridges.
Brichoe, Che. brittle.
Bricken, bridle up the head.
Bridgenorth, (for Burgmor f.) a Town in Shropſhire.
Brie, Brieze, horſe-fly, gad-bee.
Brief, as Breve; alſo two full times (in Muſick.)
Briewr, as Bruyere.
Brigd, f. debate.
Brigade, -do, f. three ſquadrons of Soldiers, 1512 men.
Brigand, f. a robber, a footman ſerving with a
Brigandine, a Coat of male; alſo as
Brigantine, a ſwift pinnace.
Brigantes, the Northern people of England.
Brigidians, Fryers and Nuns of the Order of
Brigidia, Brigit, Bride, a Princeſs of Swedeland; alſo an Iriſh Saint.
Brik, o. narrow, ſtrait.
Brilliant, f. glittering.
Brime, bring, Sf.
Bringer-up, the laſt man of a file.
Brinne, o. burn.
Brione, wild-vine.
Briſeis, Achilles's Miſtreſs.
Brise [the hops] ſhatter.
Brittannia, This Iſland of England, Wales, and Scotland, from
Brith, Br. painted.
Briſtol, -ow, a City partly in Sommerſet, and partly in Gloceſter-ſhire.
Britomartis, a Cretan Lady Inventreſs of hunting-nets.
Britonner, o. a bragger, boaſter.
Brize, as Breez.
Broach, the next ſtart above the beam-antler of a ſtag.
Brocade, Sp. cloth mixt with Gold or Silver.
Broccarii, ſc. Mediators in any buſineſs.
Brochity, crookedneſs [of Teeth.]
Brocket, pricket, ſpitter, a red Deer two years old.
Broch, Brooch, o. a picked ornament [of Gold.]
Brock, badger.
Rock-

The structure of the giant component



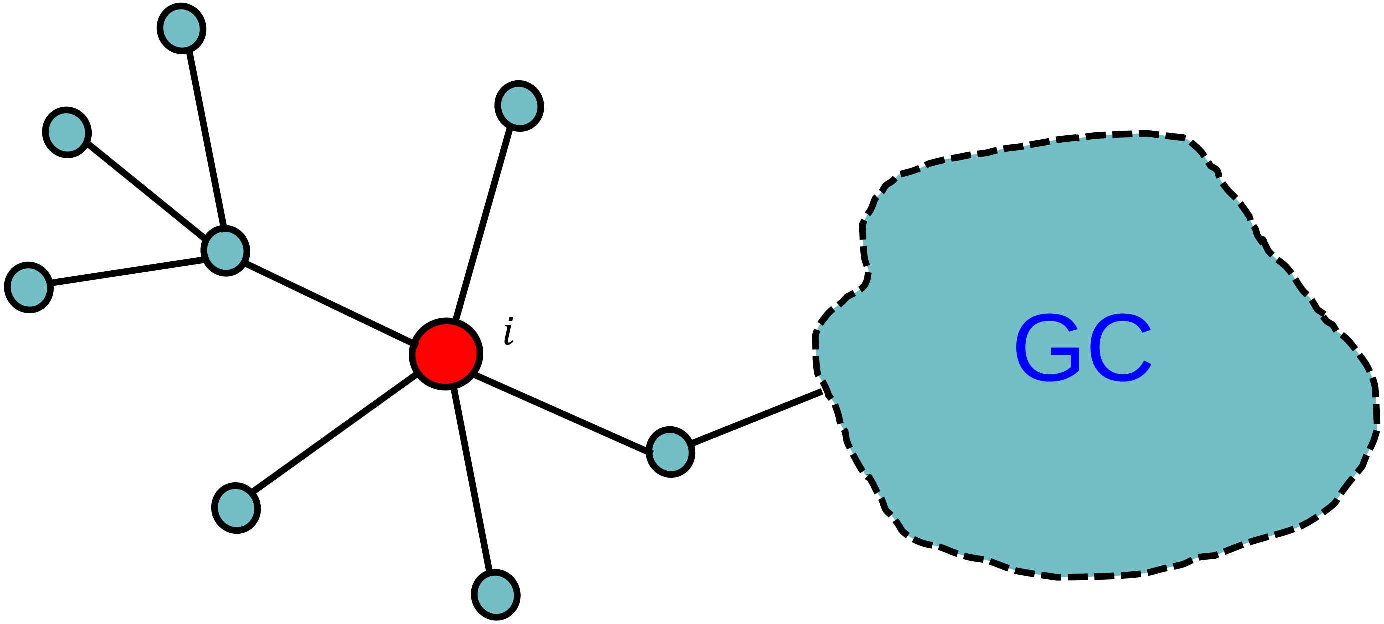
ER network

On the giant component there are degree-degree correlations



Statistical analysis of articulation points

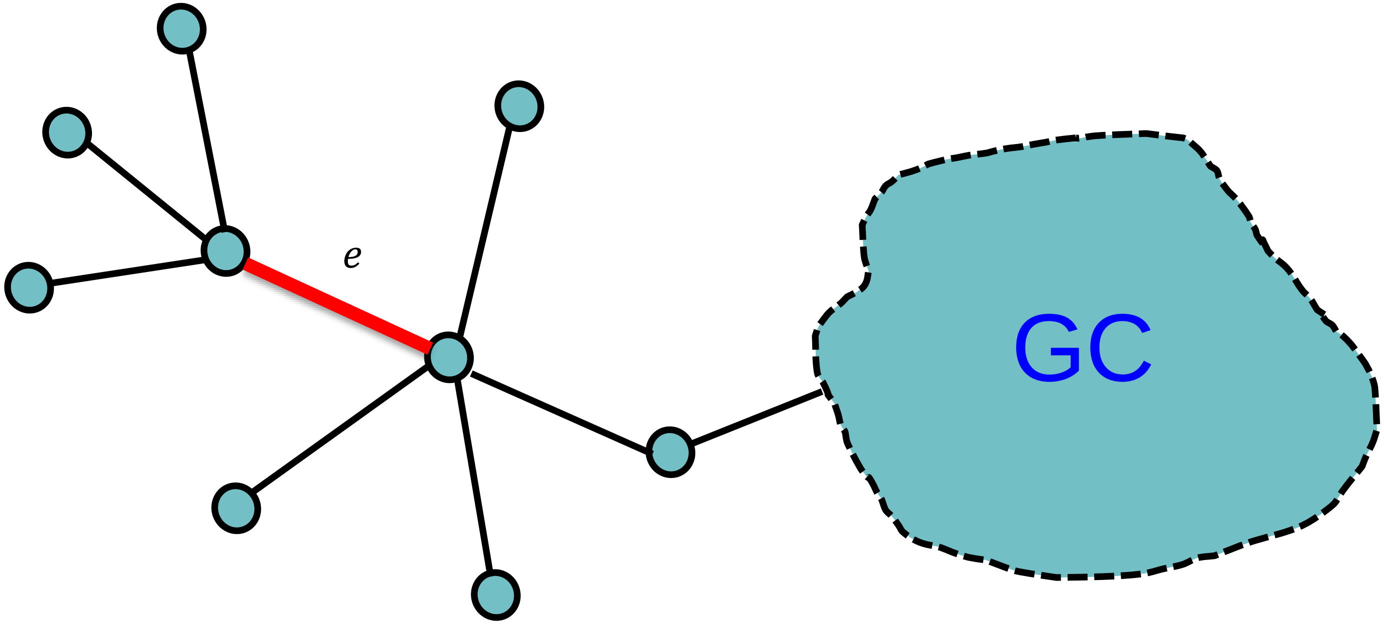
$$P(i \in \text{GC}) = g$$



I. Tishby, O. Biham, R. Kühn and E. Katzav, Statistical analysis of articulation points in configuration model networks, *Phys. Rev.* E **98**, 062301 (2018)

Statistical analysis of bredges

$$P(e \in \mathrm{GC}) = (2 - \tilde{g}\,)\tilde{g}$$



H. Bonneau, O. Biham, R. Kuhn and E. Katzav, Statistical analysis of edges and bredges in configuration model networks, *Phys. Rev.* E **102**, 012314 (2020)

Articulation Points

Using g and $\tilde{g}$ we derive the probability that a random node is an AP.

Summarizing the 3 cases:

$$P(i \in AP \mid k) = \begin{cases} 0, & k = 0,1 \\ (1 - \tilde{g}^k), & k \geq 2 \end{cases}$$

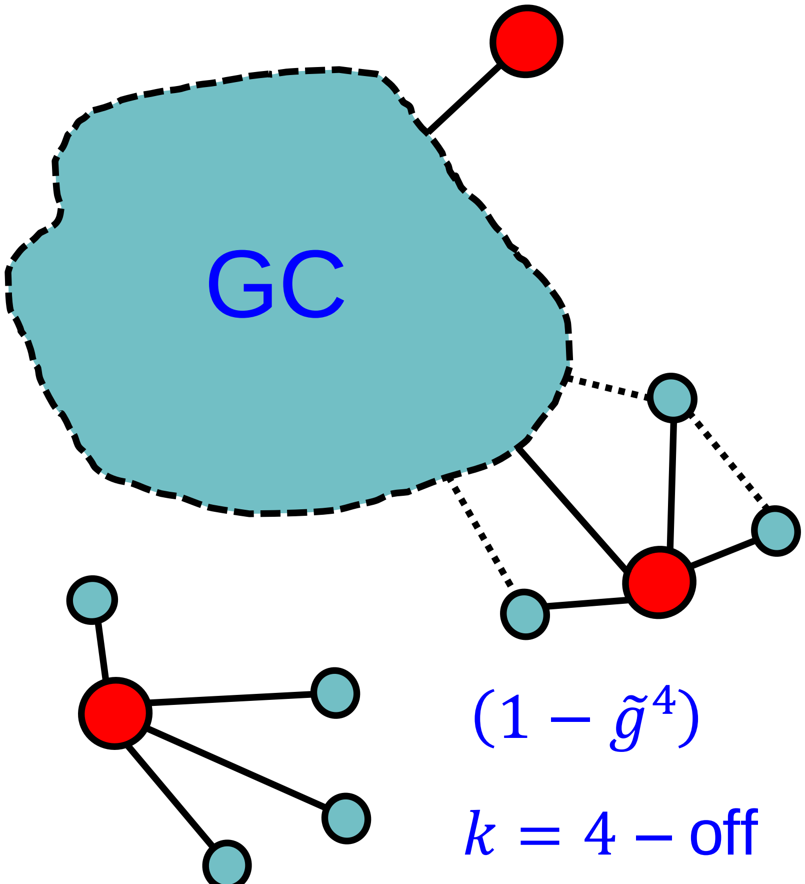
$$\Downarrow$$

$$P(i \in AP) = \sum_{k \geq 2} (1 - \tilde{g}^k) P(K = k)$$

Use Bayes' theorem:

$$P(k \mid AP) = \frac{P(i \in AP \mid k)}{P(i \in AP)} P(k)$$

$k = 1$ – not an AP



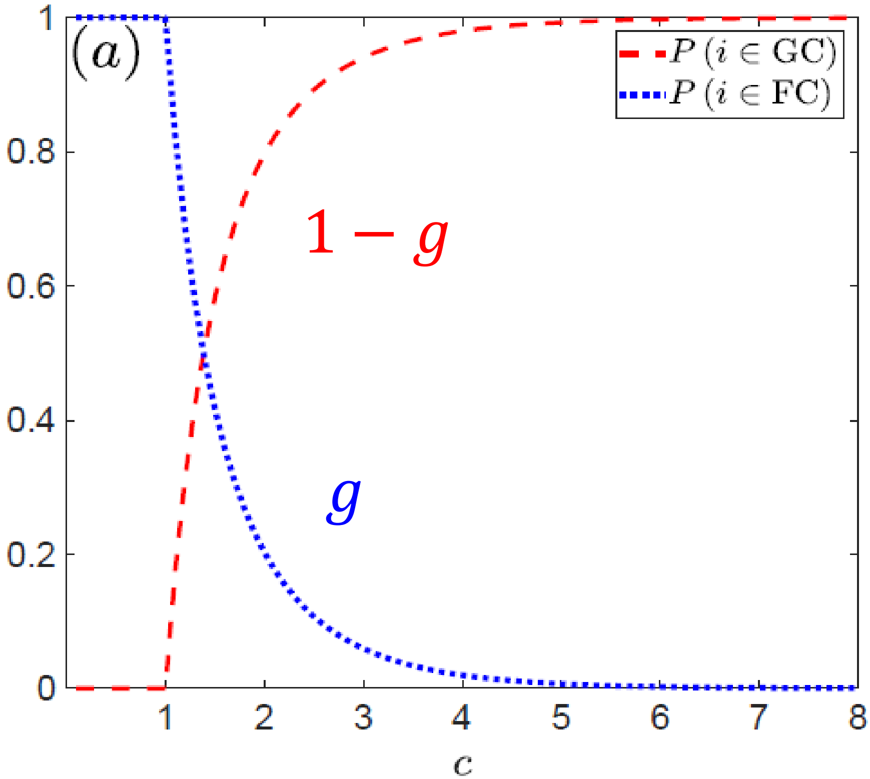
$(1 - \tilde{g}^4)$

$k = 4$ – off or on GC

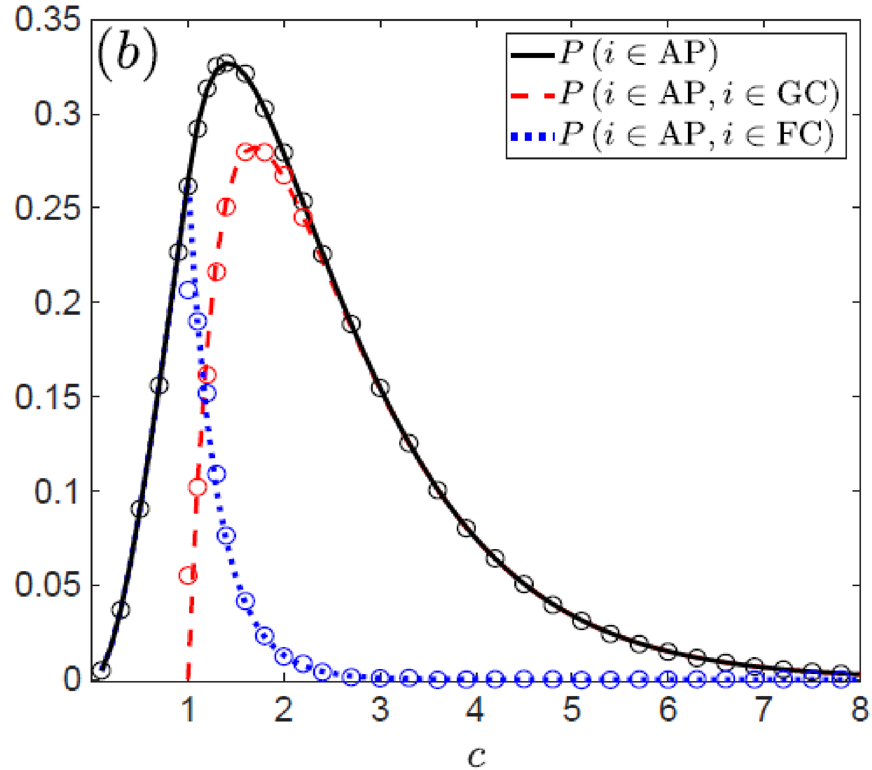
Articulation Points - Results

For example for ER:

$$P(i \in GC) \equiv g$$



$$P(i \in AP)$$



Articulation Points

Using g and $\tilde{g}$ we can also derive the probability that an AP has a given articulation rank r and the network's mean articulation rank $\langle R \rangle$:

$$P(R = r \mid FC) = \begin{cases} \dfrac{1}{1-g} P(K=0) + \dfrac{1-\tilde{g}}{1-g} P(K=1), & r = 0 \\ \dfrac{(1-\tilde{g})^{r+1}}{1-g} P(K = r+1), & r \geq 1 \end{cases}$$

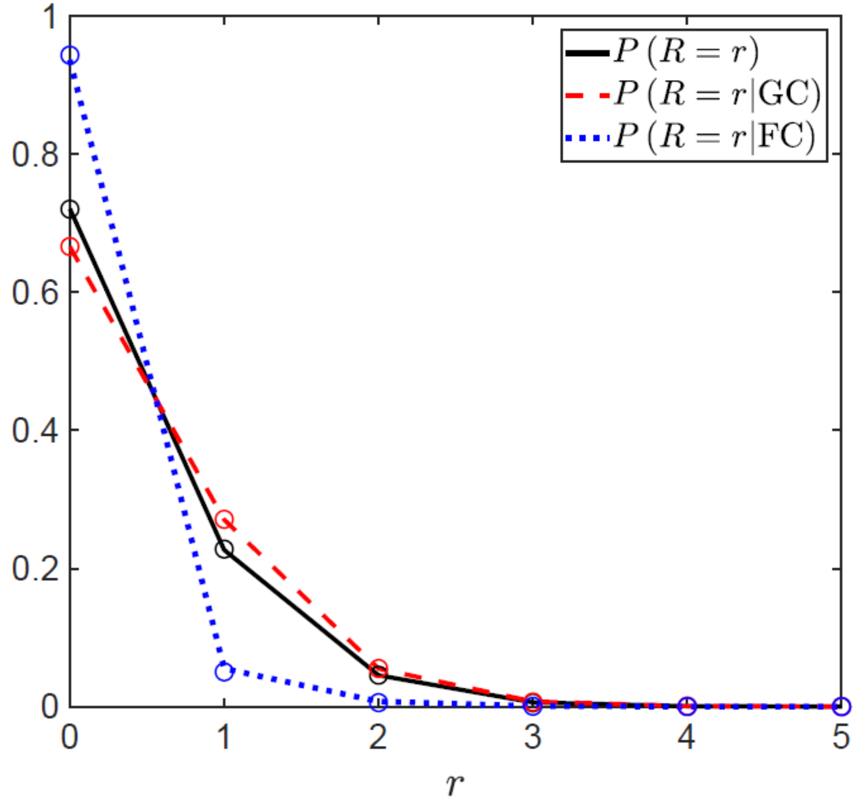
$$P(R = r \mid GC) = \frac{(1-\tilde{g})^r}{g} \sum_{k \geq r+1} \binom{n}{k} \tilde{g}^{k-r} P(K = k)$$

$$\langle R \rangle = (1-\tilde{g})\langle K \rangle + (1-g) + P(K=0)$$

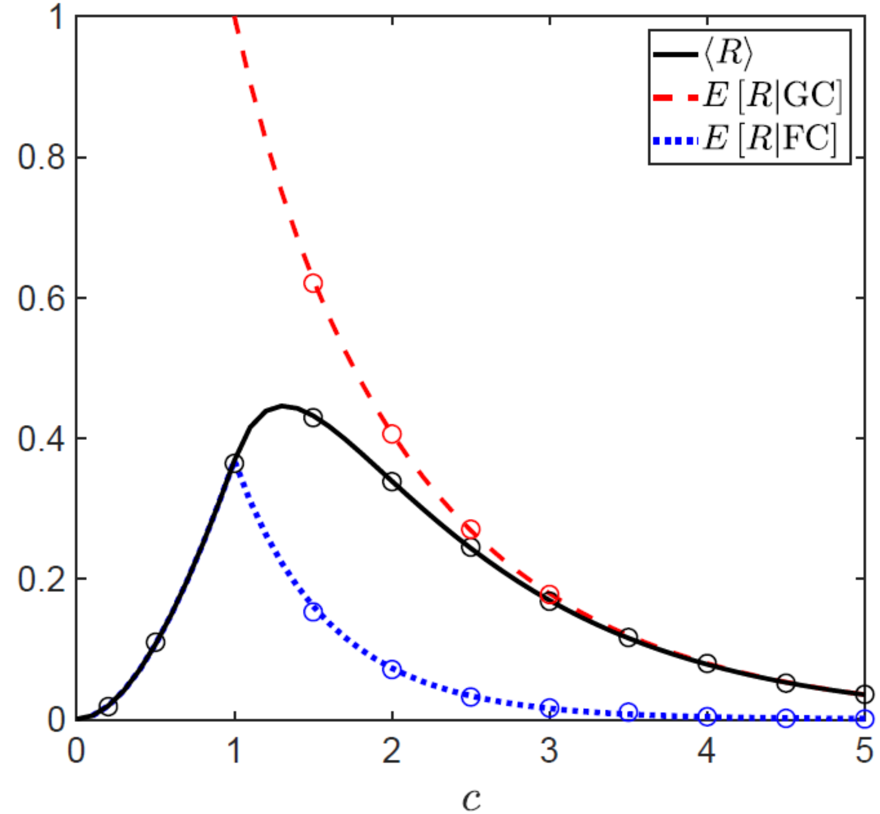
Articulation Points - Results

For example for ER:

$P(R = r)$



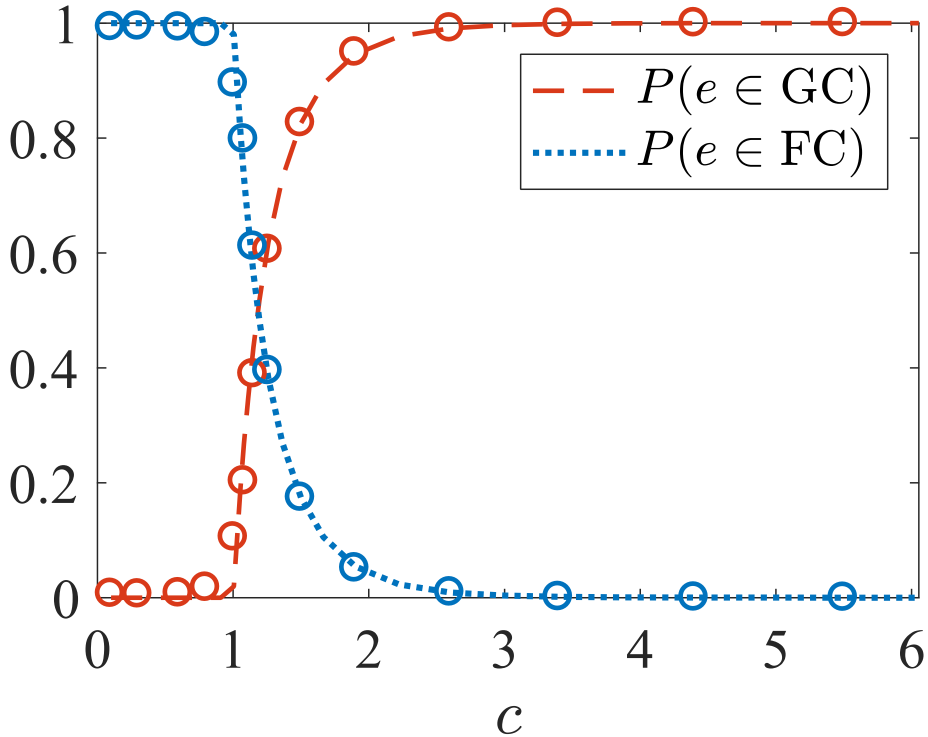
$\langle R \rangle$



The fraction of edges that reside on the giant component

$$P(e \in \mathrm{GC}) = (2 - \tilde{g}\,)\tilde{g}$$

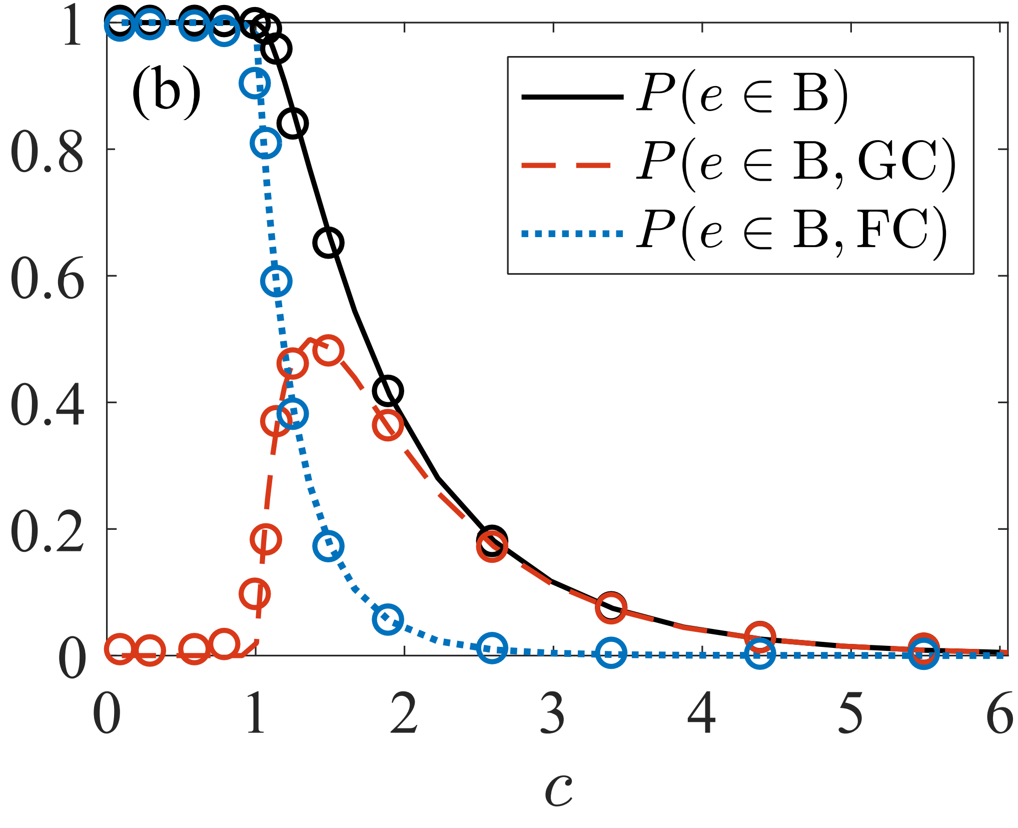
ER network



The probability that a random edge is a bredge

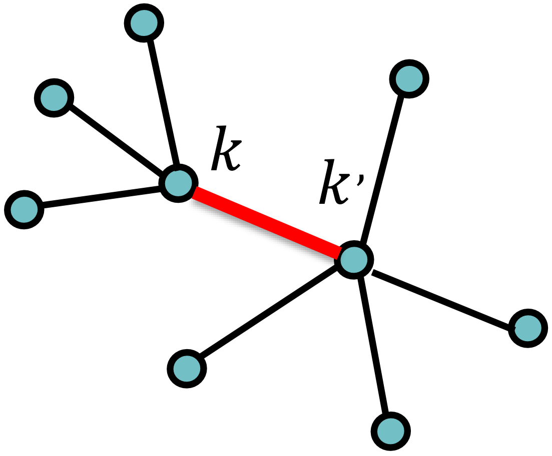
$P(e \in \mathrm{B}) = 1 - \tilde{g}^2$

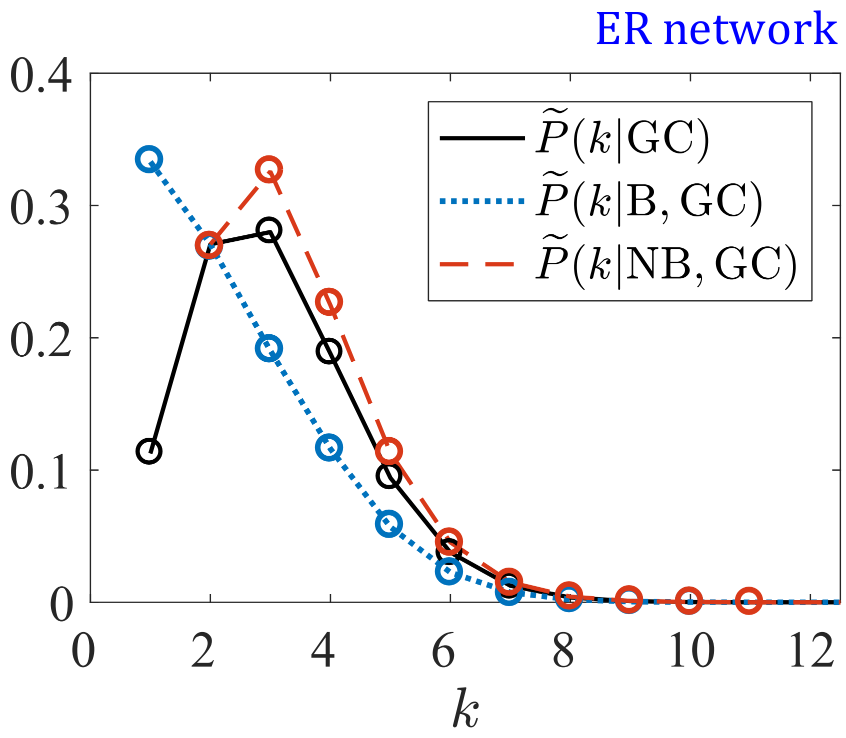
ER network



The degree distributions of end-nodes of edges and bredges

$$P(k|\mathrm{GC}) = \frac{1-(1-\tilde{g})^k}{g} P(k)$$

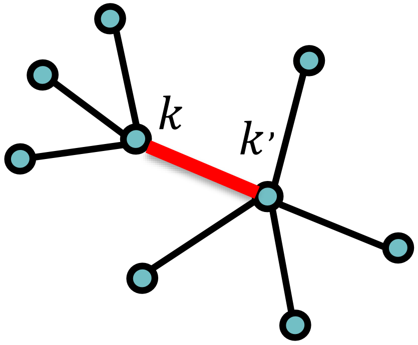


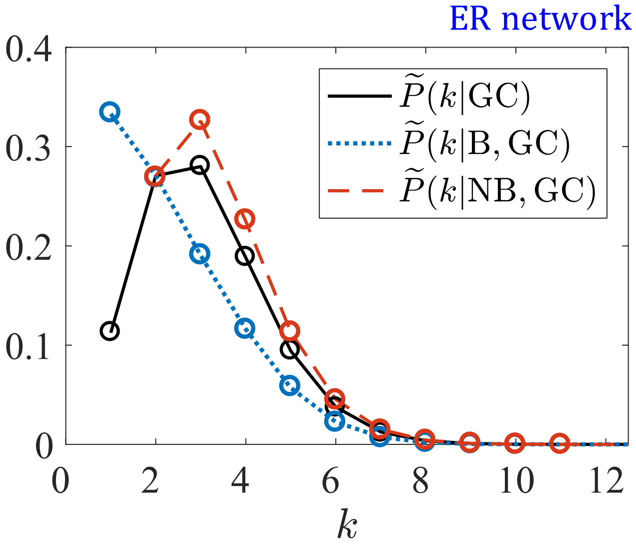


The degree distributions of end-nodes of edges and bredges

$$P(k|\mathrm{GC}) = \frac{1-(1-\tilde{g})^k}{g} P(k)$$

$$P(k|\mathrm{B,GC}) = \frac{1+(2\tilde{g}-1)(1-\tilde{g})^{k-2}}{2\tilde{g}} P(k)$$

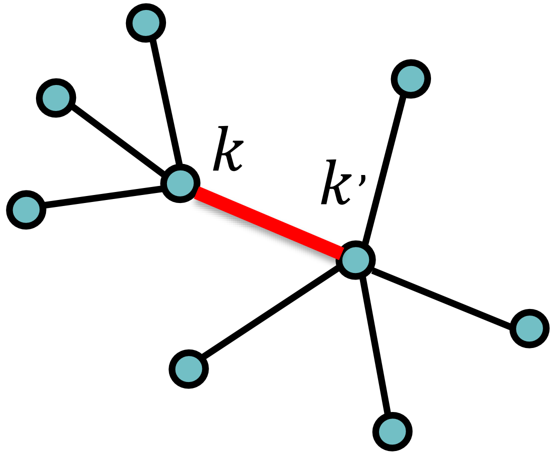




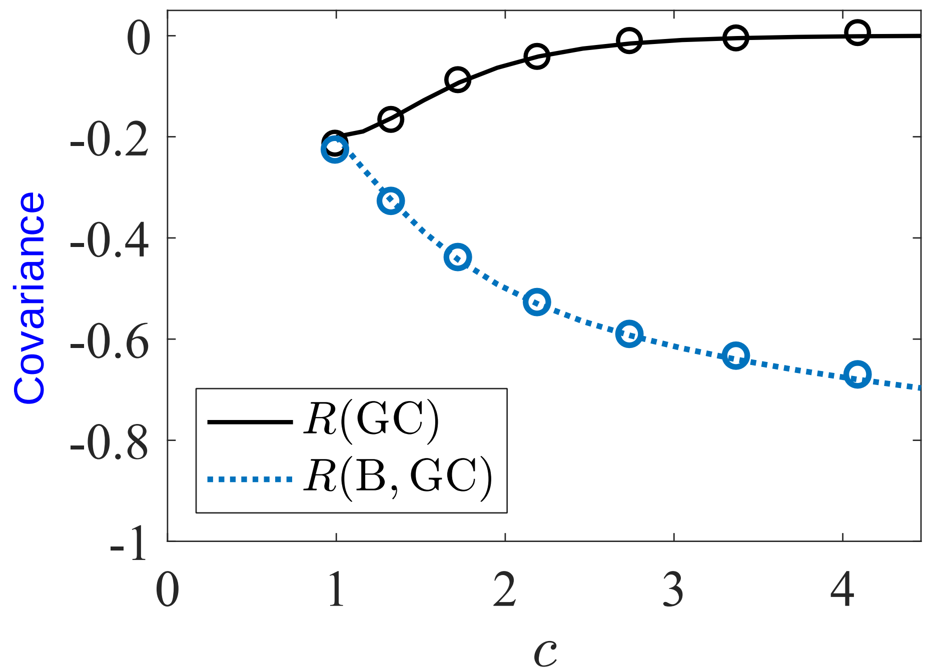
The degree-degree correlations between end-nodes

Covariance: $$R = \frac{\mathbf{E}[KK'] - \mathbf{E}[K]\mathbf{E}[K']}{\mathbf{V}[K]}$$

Degree-degree correlations are negative and concentrated in the bredges: The giant component is disassortative. Moreover, the bredges account for all the disassortativity in the giant component.

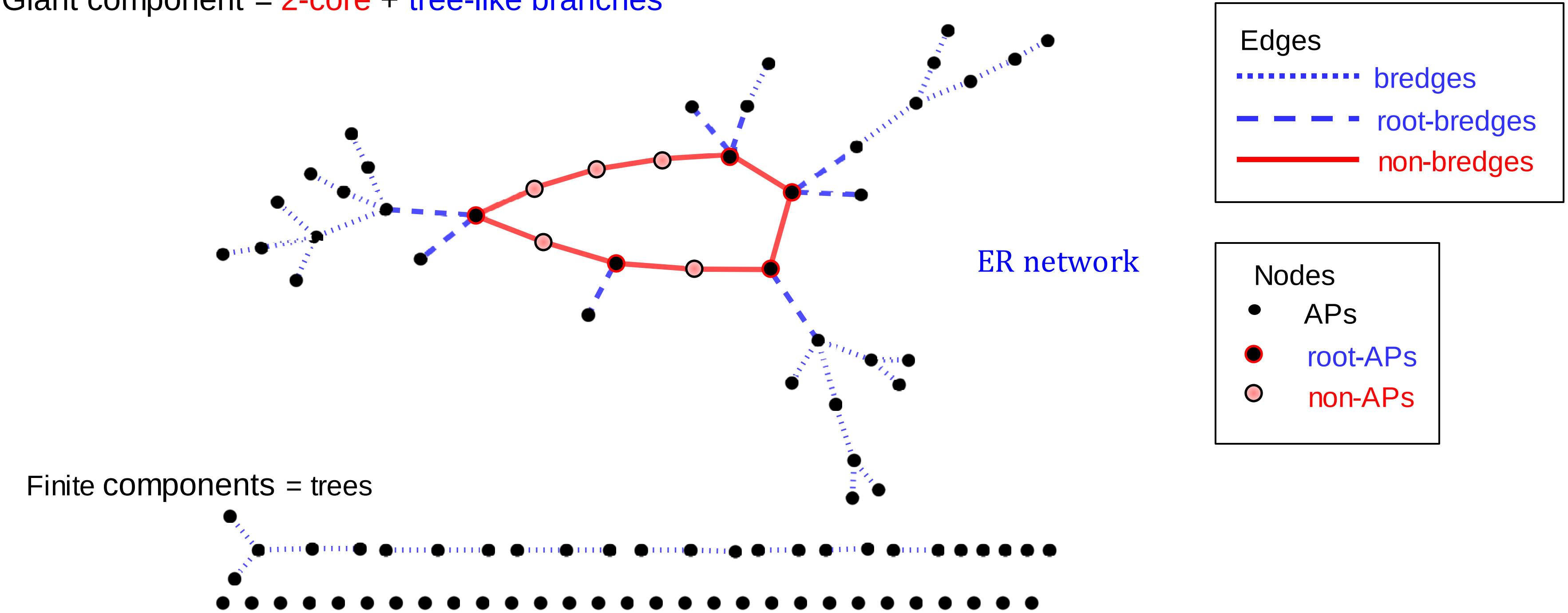


ER network



APs/Bredges and the geometry of networks

Giant component = 2-core + tree-like branches

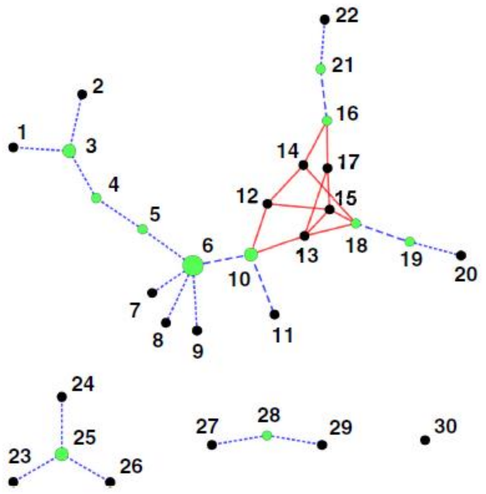


APs and Bredges are fundamental objects in the geometry of networks,
and play an important role in failure and attacks, epidemic spreading etc…

Statistical analysis of Bredges

Using the cavity method:

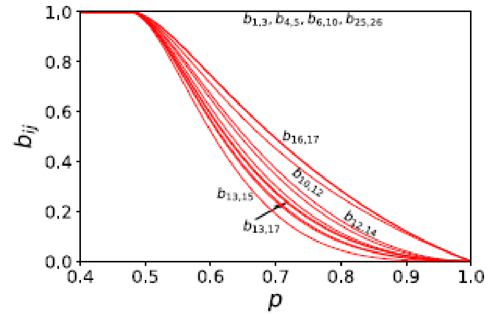
- The probability that a random edge is a bredge $P(e \in B)$
- The degree-distribution of end-nodes of bredges $P(k, k'|B)$
- The degree-degree correlation between end-nodes of bredges
- …



We also studied the statistics of APs and bredges during a bond percolation process that retains a fraction p of the edges in the network, questions like:

What is the probability for a given edge to become a bredge after deleting a fraction of the edges from the network (even if it wasn't a bredge initially)

Can be helpful to design vaccination or attack strategies



H. Bonneau, I. Tishby, O. Biham, E. Katzav and R. Kühn, Fate of articulation points and bredges in percolation, Phys. Rev. E **103**, 042302 (2021)

Summary

- The giant component consists of a 2-core and tree-like branches
- Some of the nodes on the giant component are APs
- The edges on the tree-like branches are bredges
- APs and bredges exhibit special statistical properties
- We obtained analytical results for:
    - The probability that a random edge is a bredge
    - The degree-distribution of end-nodes of bredges
    - The degree-degree correlation between end-nodes
    - …
- It was found that the degree-degree correlations on the GC are negative
- Moreover, these negative correlations are concentrated on the bredges
- These negative correlations are crucial for the integrity of the GC