Glassy random walks: Large deviations and aging

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Overview

- Two broad classes of non-equilibrium dynamics: driven and aging
- One general driving mechanism: trajectory biasing
- Directly linked to dynamical phase transitions and large deviations
- How do driving and aging interact?
- Probe in models of slow dynamics on networks: glassy random walks

The Kühn connection

Method for finding dynamical free energies:

- Susca, Vivo & Kühn (2019):
  Top eigenpair statistics for weighted sparse graphs

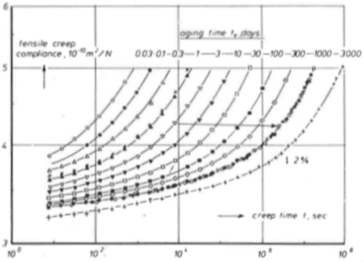
Physics of localization transition in large deviations:

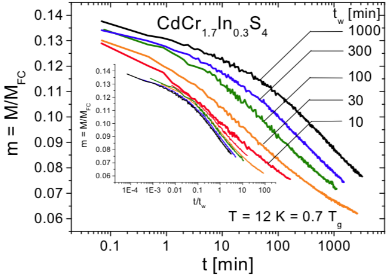
- Bacco, Guggiola, Kühn & Paga (2016):
  Rare events statistics of random walks on networks: localisation and other dynamical phase transitions

Outline

Aging

Occurs not just in living systems. . .



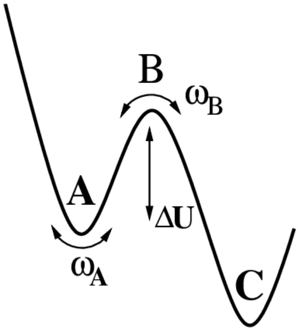


- Aging systems could reach equilibrium – but are too slow
- Significant dependence of properties on age since preparation
- Polymers, spin glasses, . . .

Simple example of aging: coarsening

- Phase separation after quench from high T
- Properties governed by growing domain size $L(t)$
- E.g. two-time correlation functions decay with ratio of L's

Aging requires complex dynamics

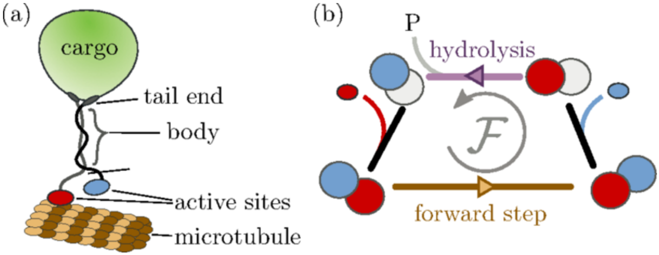


- Contrast with escape from single metastable state
- Beyond age $\sim$ metastable lifetime, age-dependences disappear
- Aging requires many states, broad spectrum of lifetimes

Outline

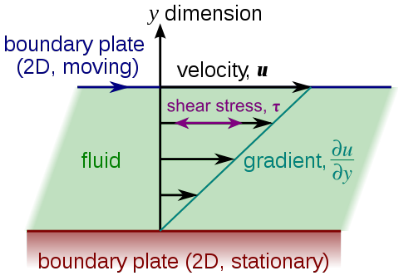
Driven systems

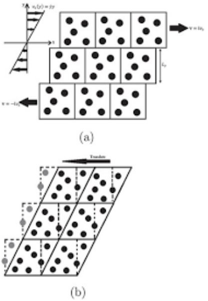
- Break detailed balance
- E.g. sheared fluids, all living systems: energy input, dissipation
- Probability currents in steady state



Modelling driven systems

- Lack of detailed balance:
  More freedom to choose parameters – underconstrained?
- Is there a systematic way of assigning free parameters?
- E.g. for motion in bulk of sheared fluid

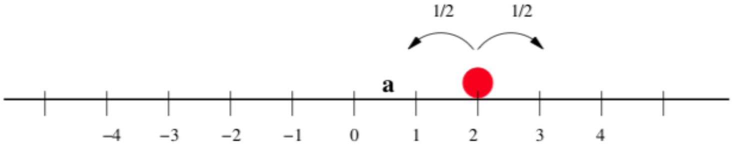




Biased trajectory ensembles

Ruelle, Spohn, Evans, ...

- Start from equilibrium dynamics
- Main idea: think of this as a distribution over trajectories
- Modify distribution to get e.g. some average current $\mathcal{A}_t$
- Which trajectory distribution has maximum entropy? (relative to equilibrium dynamics)



Biased trajectory ensembles – cont.

- Maximum entropy problem analogous to equilibrium statistical mechanics:
  Constraints on averages give exponential weight factors
- E.g. Boltzmann distribution constrains $\langle E \rangle$, gives weight $e^{-\beta E}$
  Normalization defines free energy f
- Similarly max ent trajectory ensemble:
  Equilibrium trajectory distribution biased by factor $\mathrm{e}^{-g\mathcal{A}_t}$
  Normalization defines a dynamical free energy $\psi(g)$
  Legendre transform $\Rightarrow$ large deviations $P(\mathcal{A}_t) \sim e^{-t\,\phi(\mathcal{A}_t/t)}$

Summary

A systematic way of describing driven systems is given by trajectory thermodynamics

Outline

Biasing trajectory probabilities

- Trajectory π; bias probability to give large/small values of $\mathcal{A}_t$:

$$P[\pi, g] = Z(g,t)^{-1} P[\pi, 0] e^{-g\mathcal{A}_t}$$

- Bias parameter g: analog of magnetic field h
- Dynamical free energy: defined by analogy with equilibrium free energy

$$\psi(g) \equiv t^{-1} \ln Z(g,t)$$

- Derivatives give cumulants, e.g.

$$-\psi'(g) = t^{-1} \langle \mathcal{A}_t \rangle$$

Setting: Stochastic dynamics

Markov chain

- Consider stochastic model with configurations $\mathcal{C}$
- Transition rates $W(\mathcal{C} \to \mathcal{C}')$
- Escape rate from $\mathcal{C}$: $r(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} W(\mathcal{C} \to \mathcal{C}')$
- Bias in a quantity measuring transitions that system makes: if configuration sequence is $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_K$

$$\mathcal{A}_t = \sum_{k=0}^{K-1} \alpha(\mathcal{C}_k, \mathcal{C}_{k+1})$$

- $\mathcal{A}_t$ = total nr. of transitions if $\alpha(\mathcal{C}, \mathcal{C}') = 1$ for all $\mathcal{C} \neq \mathcal{C}'$ (activity)
- Or $\alpha(\mathcal{C}, \mathcal{C}')$ could measure contribution of $\mathcal{C} \to \mathcal{C}'$ to total current, accumulated shear strain, entropy current, ...

Biased & auxiliary master operators

- Dynamical partition function is largest eigenvalue of biased master operator $\mathbb{W}(g)$ with elements

$$\langle\mathcal{C}|\mathbb{W}(g)|\mathcal{C}'\rangle = \begin{cases} W(\mathcal{C}' \to \mathcal{C})\mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})}, & \mathcal{C} \neq \mathcal{C}' \\ -r(\mathcal{C}), & \mathcal{C} = \mathcal{C}' \end{cases}$$

- This does not conserve probability
- But can restore this by defining effective rates (Jack & PS, Chetrite & Touchette)

$$W^{\mathrm{aux}}(\mathcal{C}' \to \mathcal{C}) = W(\mathcal{C}' \to \mathcal{C})\mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})}\frac{u_\tau(\mathcal{C})}{u_\tau(\mathcal{C}')}$$

- Metropolis-like factor $\exp\{-\beta[E_\tau^{\mathrm{eff}}(\mathcal{C}) - E_\tau^{\mathrm{eff}}(\mathcal{C}')]/2\}$, with effective potential

$$E_\tau^{\mathrm{eff}}(\mathcal{C}) = (-2/\beta)\ln u_\tau(\mathcal{C})$$

Outline

Trap models

- Picture of glassy dynamics: at low T have activated jumps...
- ...between local energy minima in configuration space
- Take each minimum as a configuration $\mathcal{C}_i$ or "trap"
- Trap depth $E_i > 0$
- Simplest assumption on kinetics gives Bouchaud trap model

$$W(\mathcal{C}_i \to \mathcal{C}_j) = \frac{1}{N} \exp(-\beta E_i)$$

  where $N =$ number of configurations
- Golf course landscape: always activate to "top" $(E = 0)$
- Mean field connectivity

Glass transition and aging

- Model specified by distribution of energies $\rho(E)$
- Typically taken as $\rho(E) = \exp(-E)$, exponential tail
- Gibbs-Boltzmann equilibrium distribution $\propto \exp(\beta E)\exp(-E)$ normalizable only for $\beta < 1$
- Glass transition at $T = 1/\beta = 1$
- For $T < 1$ system must age, typical $E \sim T\ln(t)$

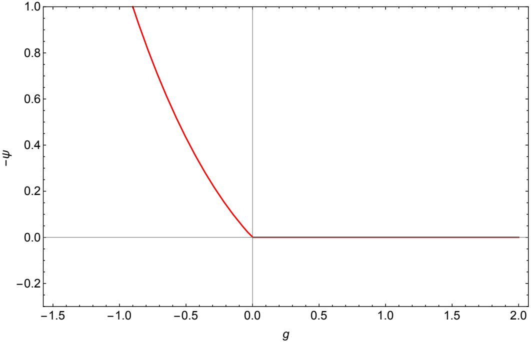
Focus

How do aging and driving (activity bias) interact?

Method: Laplace transforms,
then look at large $t - \tau$ or τ $(z \to 0)$

Dynamical free energy

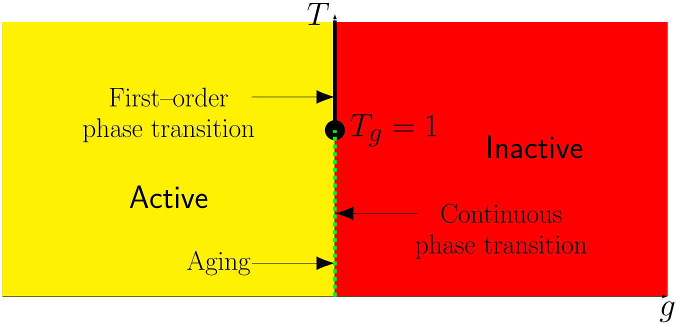
$T = 2.5$



Dynamical phase transition, active to inactive

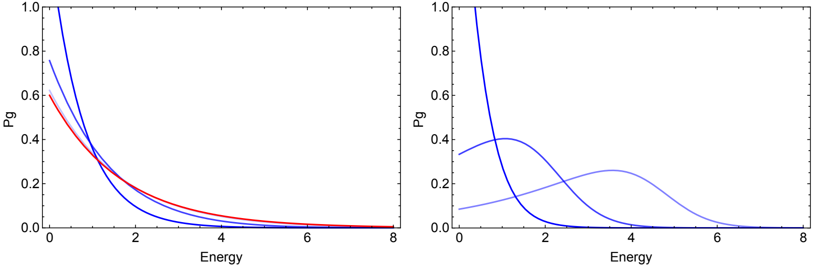
Reminder: $-\psi'(g) =$ average activity

Phase diagram



Above average activity: active phase

$g = -2$ (dark), -0.2, -0.02 (light), steady state trap depth distributions

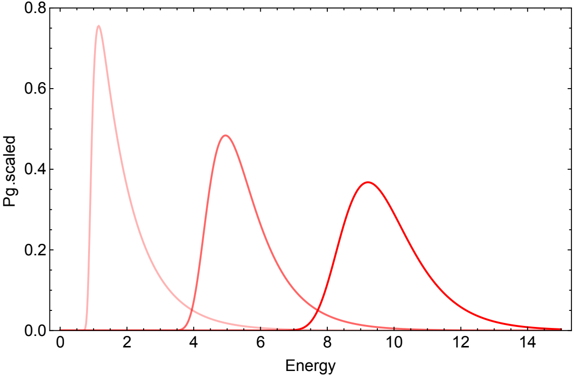


Left: $T = 2.5$; right: $T = 0.7$

For $T < 1$, typical trap depth increases as $g \to 0$;
remnant of transition to aging dynamics
Effective potential $E^{\text{eff}} = (2/\beta)\ln(1 + \psi e^{\beta E})$

Below average activity: inactive phase

$g > 0$, large t, $p_0(E) = \rho(E)$, $T = 0.1, 0.5, 1.0$ left to right



$p_\tau(E) \propto \rho(E)\exp(-te^{-\beta E})$ (away from boundaries)

Independent of g and τ

$E^{\text{eff}} = 2T(t-\tau)e^{-\beta E}$ is time-dependent

Outline

BM model basics

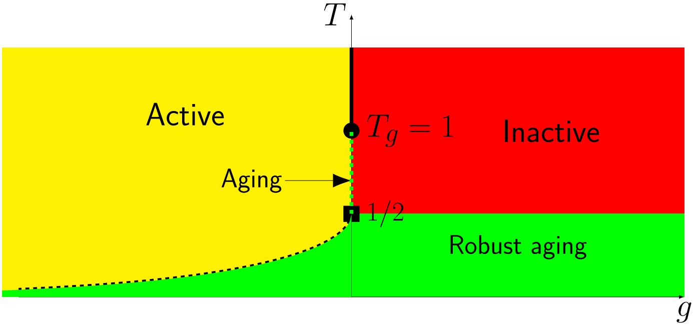
- Like Bouchaud model but Glauber rates:

$$W(\mathcal{C}_i \to \mathcal{C}_j) = \frac{1}{N}\frac{1}{1+\exp[\beta(E_i - E_j)]}$$

- Same equilibrium state, same glass transition temperature
- Aging different: entropic aging at low T, running out of lower energy states
- Dynamics not frozen even at $T = 0$

Phase diagram

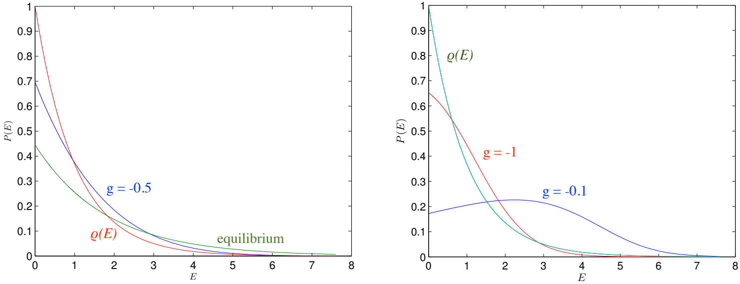
Analytical prediction, confirmed numerically by finite-size scaling



- Qualitative change at $T = 1/2$
- $T > 1/2$: shows Bouchaud-like behaviour, can be confirmed by explicit coarse-graining (Cammarota & Marinari)
- $T < 1/2$: qualitatively different, mainly downward jumps

Trap depth distributions in active phase

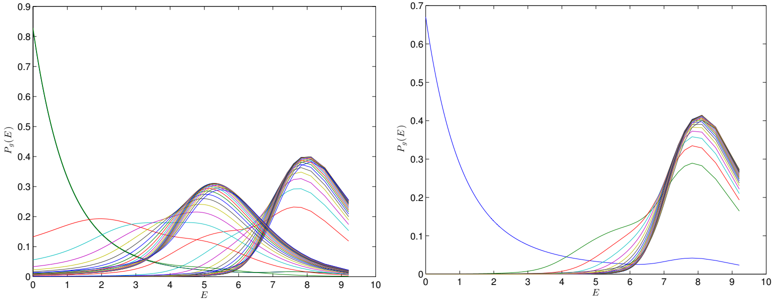
$T = 1.5$, $g = -0.5$ (left), $T = 0.8$, $g = -0.1, -1$ (right)



- For $T < 1$, distributions again shift to large E on approaching inactive phase

Inactive phase, $T > 1/2$

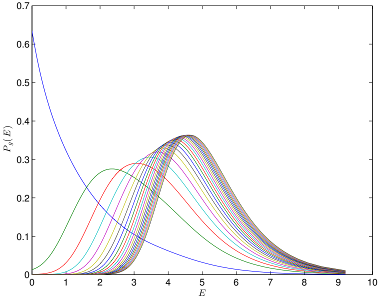
$T = 0.8$, $g = 0.25$, $t = 100$ and 1000 (left), $g = 0.5$, $t = 1000$ (right)

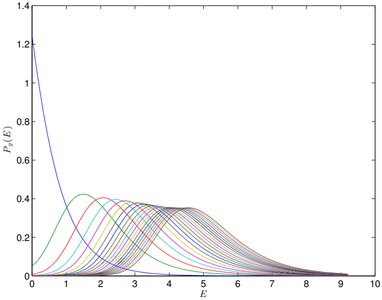


- $p_\tau(E)$ for increasing τ approaches shape only dependent on t
- System rapidly descends to deep traps
- Total nr. of jumps finite, average activity decays as $\tau^{-1-\alpha}$

Inactive phase, $T < 1/2$

$T = 0.2$, $g = 0.5$, $t = 1000$ (left) and $g = -0.5$ (right)



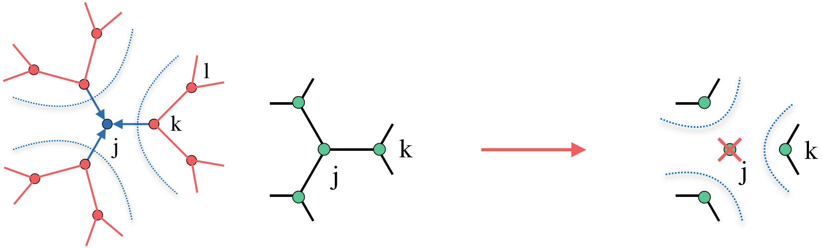


- Aging persists in presence of bias: "robust aging"
- Activity decays as τ^{-1}, like for $g = 0$
- Total number of jumps diverges with t

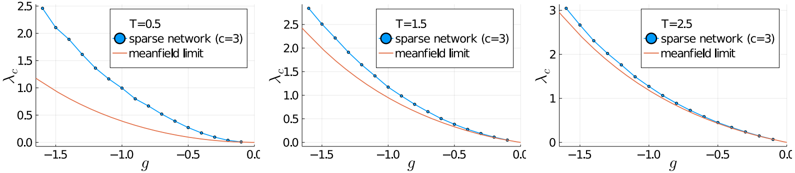
Outline

Bouchaud model on random regular network

- Fixed finite connectivity c
- Use cavity theory to find largest eigenvalue of $\mathbb{W}(g)$ and associated eigenvector
  (Kabashima, Susca et al)
- Apply to large single instances of networks
  (population dynamics subtle)

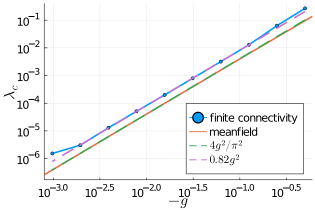


Dynamical free energy

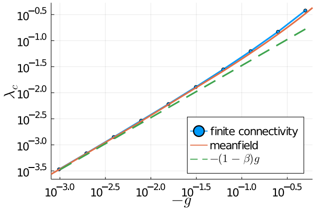


- $\psi(g)$ qualitatively similar to mean field limit $c \to \infty$
- High temperature limit can be taken in cavity equations: independent of c

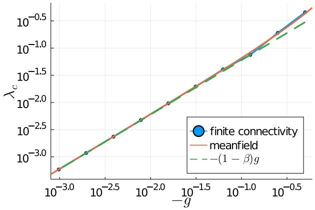
Dynamical free energy near $g = 0$



(a) $T = 0.5$



(b) $T = 1.5$

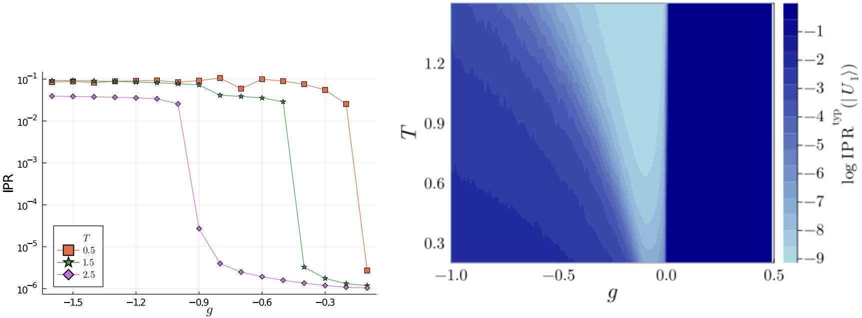


(c) $T = 2.5$

Activity $-\psi'(g)$ for $g \to 0$ independent of c (for $T > 1$)

Localization transitions & phase diagram

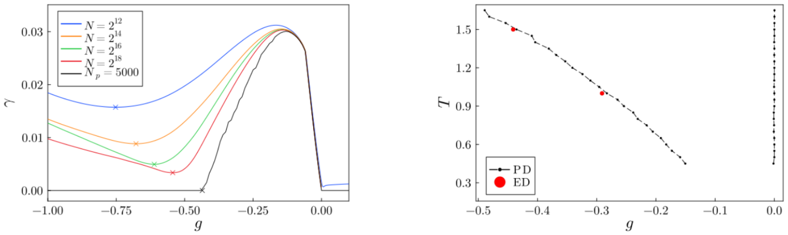
Left: single instance with $N = 2^{20}$ nodes



- Dynamical transition at $g = 0$ is always a localization transition
- Additional localization transition at T-dependent $g < 0$

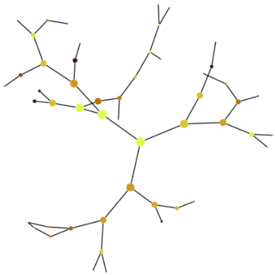
Spectral gap

Single instance vs population dynamics

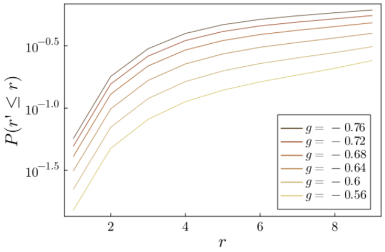


(a) spectral gap **(b)** phase transitions

Structure of localized eigenvectors for $g < 0$



(a) typical localisation cluster



(b) cumulative probability

- Localization on shallow traps
- Requires clusters of shallow traps (compare De Bacco et al)

Conclusion & Outlook

Summary

- Driving by activity bias in Bouchaud trap model has non-trivial effects
- Aging is fragile: bias towards inactivity $\Rightarrow$ freezing
- Low-activity phase: time-dependent effective potential forces time-independent $p_\tau(E)$
- Barrat-Mézard: qualitatively different for $T < 1/2$
- Aging is robust to biasing towards inactivity

Outlook

- Universality classes of aging (robust, fragile, ...)?
- Aging in "directly" driven systems?
- Nature & dynamical consequences of localization transitions

Link to large deviations

- E.g. in Ising model magnetization distribution

$$P(M) \sim e^{-N\phi(m)}$$

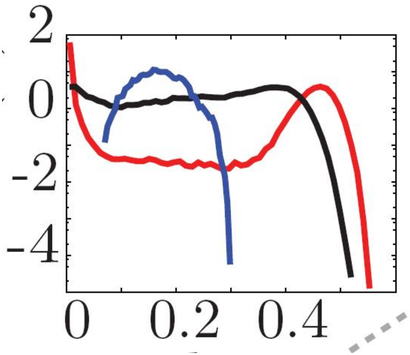
- Large deviation function $\phi(m)$, with $m = M/N$
- Free energy as function of magnetic field h

$$f(h) = -N^{-1} \ln\langle e^{hM} \rangle \approx \min_m \phi(m) - hm$$

- So Legendre transform links $\phi(m)$ and $f(h)$: change of ensemble, fixed m vs fixed h
- Works the same for trajectories: can get $P(\mathcal{A}_t) \sim e^{-t\,\phi(\mathcal{A}_t/t)}$ from dynamical free energy ψ

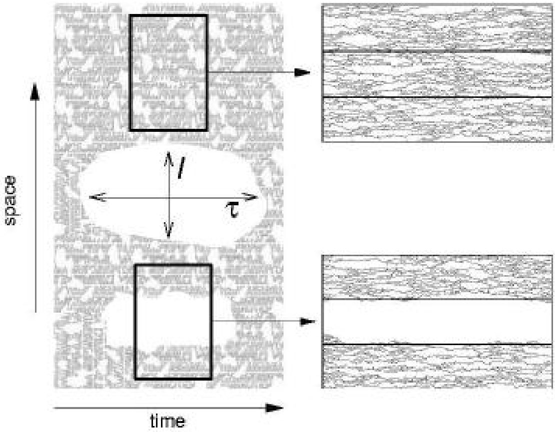
Example: Distribution of total activity

Spin model with constrained kinetics



- $\mathcal{A}_t$ = total number of transitions (spin flips)
- Two peaks in $\ln P(\mathcal{A}_t)$: phase coexistence
- Analogous to magnetization in Ising model at $h = 0$

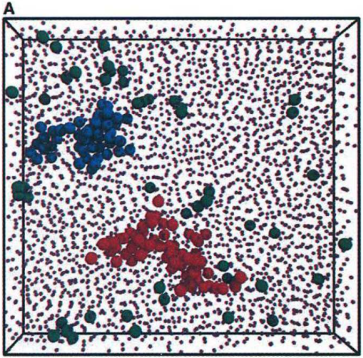
Space-time plots: Dynamical heterogeneity

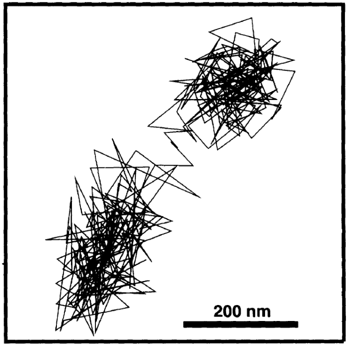


Domains of different space-time phases
(Jack, Garrahan, Chandler, Lecomte, van Wijland, Lecomte, Pitard, . . .)

Dynamical heterogeneity in colloidal glasses

(Eric Weeks group)

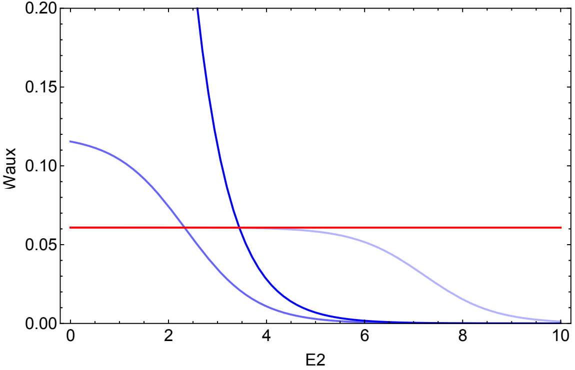




Dynamical heterogeneity makes individual particle motion intermittent

Bouchaud model: Effective transition rates

$g = -2, -0.2, -0.02$



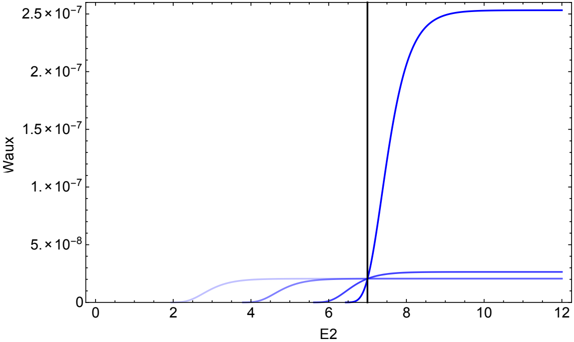
$W^{\mathrm{aux}}(E_1 \to E_2)$ (for $E_1 = 2$, $T = 0.7$)

Jumps to shallow traps are favoured

Overall rate increases with $|g|$

Bouchaud model: Effective transition rates

$t - \tau = 10^3$(light), $10^4, 10^5, 10^6$(dark)



$$W^{\text{aux}}(E_1 \to E_2) \text{ (at } E_1 = 7, T = 0.4)$$

At early times jumps only into deep traps

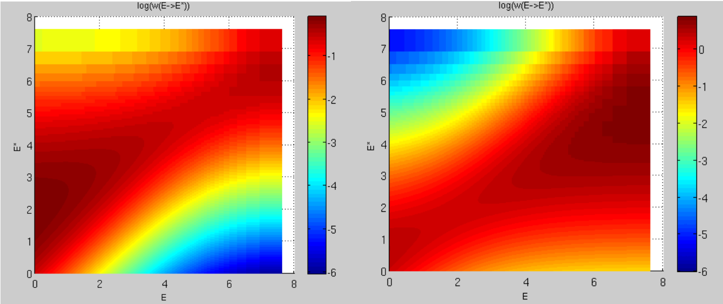
Effective threshold level rises towards end of trajectory

Time-dependent activity

- Average activity now depends on time τ along trajectory
- Goes as τ^{-1-T} (away from temporal boundaries)
- Jumps concentrated in initial part of trajectory (for $T < 1$)
- Total activity is $\propto (e^g - 1)^{-1}$, only finite number of jumps
- Bias towards inactivity freezes system

BM model: Effective transition rates in active phase

$T = 0.8$, $g = -0.1$ (left), $T = 0.8$, $g = -1$ (right)



- Jumps biased towards more shallow traps
- Resulting rates are non-monotonic in arrival trap depth

Stochastic dynamics

Markov, unbiased

- Start from stochastic model with configurations $\mathcal{C}$
- Transition rates $W(\mathcal{C}' \to \mathcal{C})$
- Master equation:

$$\frac{\partial}{\partial t} p(\mathcal{C}, t) = -r(\mathcal{C}) p(\mathcal{C}, t) + \sum_{\mathcal{C}' \neq \mathcal{C}} W(\mathcal{C}' \to \mathcal{C}) p(\mathcal{C}', t)$$

- Escape rate from $\mathcal{C}$: $r(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} W(\mathcal{C} \to \mathcal{C}')$
- Matrix/vector form: let $|P(t)\rangle = \sum_{\mathcal{C}} p(\mathcal{C}, t) |\mathcal{C}\rangle$, then

$$\frac{\partial}{\partial t} |P(t)\rangle = \mathbb{W} |P(t)\rangle$$

- Master operator $\mathbb{W}$ has matrix elements $\langle \mathcal{C} | \mathbb{W} | \mathcal{C}' \rangle = W(\mathcal{C}' \to \mathcal{C}) - \delta_{\mathcal{C},\mathcal{C}'} r(\mathcal{C})$

Time-integrated quantities

- In simplest case, might want to bias trajectories according to cumulative value of some observable

$$\mathcal{B}_t = \int_0^t \mathrm{d}t' B(t')$$

where $B(t') = b(\mathcal{C}(t'))$ depends only on configuration $\mathcal{C}(t')$

- Or bias depending on transitions that system makes:
if configuration sequence is $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_K$, use

$$\mathcal{A}_t = \sum_{k=0}^{K-1} \alpha(\mathcal{C}_k, \mathcal{C}_{k+1})$$

- $\mathcal{A}_t$ = total number of moves if $\alpha(\mathcal{C}, \mathcal{C}') = 1$ for all $\mathcal{C} \neq \mathcal{C}'$ (activity)
- Or $\alpha(\mathcal{C}, \mathcal{C}')$ could measure contribution of $\mathcal{C} \to \mathcal{C}'$ to total current, accumulated shear strain, entropy current, ...

Biasing trajectory probabilities

- Trajectory π; bias probability to give large/small values of $\mathcal{B}_t$:

$$P[\pi, g] = Z(g,t)^{-1} P[\pi, 0] \exp\left[-g\mathcal{B}_t\right]$$

- Bias parameter g; canonical version of hard constraint on $\mathcal{B}_t$
- Trajectory partition function (discretize, $t = M\Delta t$)

$$Z(g,t) = \sum_{\mathcal{C}_0 \ldots \mathcal{C}_M} \exp\{\Delta t \sum_{i=1}^{M} [W(\mathcal{C}_{i-1} \to \mathcal{C}_i) - g b(\mathcal{C}_{i-1})]\} p_0(\mathcal{C}_0)$$

$$\to \langle e | \mathrm{e}^{\mathbb{W}(g)t} | 0 \rangle, \qquad \mathbb{W}(g) = \mathbb{W} - g \sum_{\mathcal{C}} b(\mathcal{C}) |\mathcal{C}\rangle\langle\mathcal{C}|$$

- Projection state $\langle e| = \sum_{\mathcal{C}} \langle \mathcal{C}|$
- Unbiased initial (e.g. steady) state $|0\rangle = \sum_{\mathcal{C}} p_0(\mathcal{C}) |\mathcal{C}\rangle$

Dynamical free energy

- Define by analogy with equilibrium free energy as

$$\psi(g) \equiv \lim_{t\to\infty} t^{-1} \ln Z(g,t)$$

- If configuration space is finite, can decompose $\mathbb{W}(g) = \sum_i \omega_i |V_i\rangle\langle U_i|$
- Then $\psi(g) = \max_i \omega_i$ (Lebowitz Spohn)
- Maximum eigenvalue "generically" non-degenerate
- Same for bias in $\mathcal{A}_t$ (activity, current etc), with

$$\langle \mathcal{C}|\mathbb{W}(g)|\mathcal{C}'\rangle = \begin{cases} W(\mathcal{C}' \to \mathcal{C})\mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})}, & \mathcal{C} \neq \mathcal{C}' \\ -r(\mathcal{C}), & \mathcal{C} = \mathcal{C}' \end{cases}$$

Bias as time-dependent master operator
(Transcribing from Chetrite & Touchette)

- Can we write biased path probability

$$P[\pi, g] = Z(g,t)^{-1} \prod_{i=1}^{M} \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}(g)\Delta t} | \mathcal{C}_{i-1} \rangle \times p_0(\mathcal{C}_0)$$

- ...as resulting from effective time-dependent master equation:

$$P[\pi, g] = \prod_{i=1}^{M} \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}_{i-1}^{\mathrm{aux}}(g)\Delta t} | \mathcal{C}_{i-1} \rangle \times p_0^{\mathrm{aux}}(\mathcal{C}_0)$$

- Idea: set

$$\langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}_{i-1}^{\mathrm{aux}}(g)\Delta t} | \mathcal{C}_{i-1} \rangle = \frac{u_i(\mathcal{C}_i)}{u_{i-1}(\mathcal{C}_{i-1})} \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}(g)\Delta t} | \mathcal{C}_{i-1} \rangle$$

Bias as time-dependent master operator (cont)

- Require: $u_M(\mathcal{C}_M) = 1$, $p_0^{\mathrm{aux}}(\mathcal{C}_0) = p_0(\mathcal{C}_0)u_0(\mathcal{C}_0)/Z(g,t)$ and normalization

$$\sum_{\mathcal{C}_i} \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}^{\mathrm{aux}}_{i-1}(g)\Delta t} | \mathcal{C}_{i-1} \rangle \equiv \sum_{C_i} \frac{u_i(\mathcal{C}_i)}{u_{i-1}(\mathcal{C}_{i-1})} \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}(g)\Delta t} | \mathcal{C}_{i-1} \rangle = 1$$

- Hence the u_i can be determined backwards in time:

$$u_{i-1}(\mathcal{C}_{i-1}) = \sum_{\mathcal{C}_i} u_i(\mathcal{C}_i) \langle \mathcal{C}_i | \mathrm{e}^{\mathbb{W}(g)\Delta t} | \mathcal{C}_{i-1} \rangle$$

- In vector notation: $\langle U_{i-1} | = \langle U_i | \mathrm{e}^{\mathbb{W}(g)\Delta t}$
- Solution: $\langle U_i | = \langle e | \mathrm{e}^{\mathbb{W}(g)(M-i)\Delta t}$
- Thus $p_0^{\mathrm{aux}}(C) = \langle e | \mathrm{e}^{\mathbb{W}(g)t} | \mathcal{C} \rangle p_0(\mathcal{C}) / \langle e | \mathrm{e}^{\mathbb{W}(g)t} | 0 \rangle$, normalized

Effective transition rates

Continuous time: $\tau = i\Delta t$, $\Delta t \to 0$

- Expanding relation between $\mathbb{W}^{\text{aux}}$ and $\mathbb{W}(g)$ to $O(\Delta t)$ gives effective rates

$$\langle \mathcal{C} | \mathbb{W}_\tau^{\text{aux}} | \mathcal{C}' \rangle = \langle \mathcal{C} | \mathbb{W}(g) | \mathcal{C}' \rangle \frac{u_\tau(\mathcal{C})}{u_\tau(\mathcal{C}')}$$

or explicitly

$$W^{\text{aux}}(\mathcal{C}' \to \mathcal{C}) = W(\mathcal{C}' \to \mathcal{C}) \mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})} \frac{u_\tau(\mathcal{C})}{u_\tau(\mathcal{C}')}$$

- Effect of $u_\tau(\mathcal{C})$ can be interpreted as Metropolis-like factor $\mathrm{e}^{-\beta[E_\tau^{\text{eff}}(\mathcal{C}) - E_\tau^{\text{eff}}(\mathcal{C}')]/2}$, with effective potential

$$E_\tau^{\text{eff}}(\mathcal{C}) = (-2/\beta) \ln u_\tau(\mathcal{C})$$

Effective exit rates

- Effective exit rates follow from normalization as

$$-\langle \mathcal{C}|\mathbb{W}^{\text{aux}}_{\tau}|\mathcal{C}\rangle = -\langle \mathcal{C}|\mathbb{W}(g)|\mathcal{C}\rangle + \frac{\langle U_\tau|\mathbb{W}(g)|\mathcal{C}\rangle}{\langle U_\tau|\mathcal{C}\rangle}$$

- Explicitly

$$r^{\text{aux}}(\mathcal{C}) = r(\mathcal{C}) + \frac{\langle U_\tau|\mathbb{W}(g)|\mathcal{C}\rangle}{\langle U_\tau|\mathcal{C}\rangle}$$

- Shift in general dependent on $\mathcal{C}$ (and τ)

Biased & auxiliary master operators

- Dynamical partition function derived from a biased master operator $\mathbb{W}(g)$ with elements

$$\langle \mathcal{C}|\mathbb{W}(g)|\mathcal{C}'\rangle = \begin{cases} W(\mathcal{C}' \to \mathcal{C})\mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})}, & \mathcal{C} \neq \mathcal{C}' \\ -r(\mathcal{C}), & \mathcal{C} = \mathcal{C}' \end{cases}$$

- This does not conserve probability
- But can restore by multiplicative reweighting (Jack & PS, Chetrite & Touchette)

$$\langle \mathcal{C}_{\tau+\Delta t}|\mathrm{e}^{\mathbb{W}^{\mathrm{aux}}_{\tau}(g)\Delta t}|\mathcal{C}_{\tau}\rangle = \frac{u_{\tau+\Delta t}(\mathcal{C}_{\tau+\Delta t})}{u_{\tau}(\mathcal{C}_{\tau})}\langle \mathcal{C}_{\tau+\Delta t}|\mathrm{e}^{\mathbb{W}(g)\Delta t}|\mathcal{C}_{\tau}\rangle$$

- Normalization forces $\langle U_\tau| = \langle e|\mathrm{e}^{\mathbb{W}(g)(t-\tau)}$

Effective transition rates

Continuous time: $\tau = i\Delta t$, $\Delta t \to 0$

- Relation between $\mathbb{W}^{\text{aux}}$ and $\mathbb{W}(g)$ gives effective rates

$$W^{\text{aux}}(\mathcal{C}' \to \mathcal{C}) = W(\mathcal{C}' \to \mathcal{C})\mathrm{e}^{-g\alpha(\mathcal{C}',\mathcal{C})}\frac{u_\tau(\mathcal{C})}{u_\tau(\mathcal{C}')}$$

- Effect of $u_\tau(\mathcal{C})$ can be interpreted as Metropolis-like factor $\mathrm{e}^{-\beta[E_\tau^{\text{eff}}(\mathcal{C})-E_\tau^{\text{eff}}(\mathcal{C}')]/2}$, with effective potential

$$E_\tau^{\text{eff}}(\mathcal{C}) = (-2/\beta)\ln u_\tau(\mathcal{C})$$

Time dependence

- Effective master operator and potential in general time-dependent
- Also state probabilities

$$p_\tau(\mathcal{C}) = \frac{\langle e|\mathrm{e}^{\mathbb{W}(g)(t-\tau)}|\mathcal{C}\rangle\langle\mathcal{C}|\mathrm{e}^{\mathbb{W}(g)\tau}|0\rangle}{Z(g,t)} = \frac{u_\tau(\mathcal{C})v_\tau(\mathcal{C})}{Z(g,t)}$$

where $|V_\tau\rangle = \mathrm{e}^{\mathbb{W}(g)\tau}|0\rangle$

- Product of forward (from past) and backward (from future) factors
- Time-dependences disappear if driven system reaches stationary state – but not if there is aging