LPTMS
Laboratoire de Physique Théorique et Modèles Statistiques

Disordered Systems Days at KCL
King's College, London — September 2023
Valentina Ros, LPTMS Orsay

Counting equilibria in high-D random systems: from Gaussian landscapes to random ecosystems

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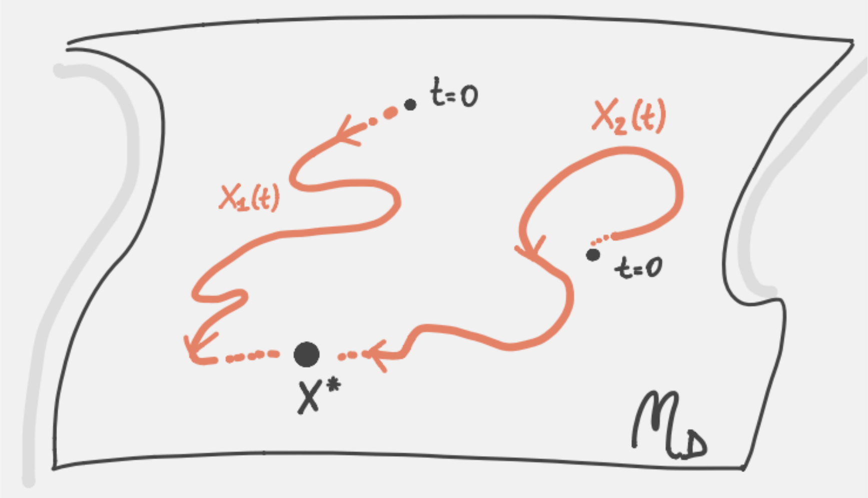
An introduction

Dynamics in high-D: many competing equilibria.

Glasses, proteins, ecosystems (microbiome), neural networks, financial markets: **many components interacting in eterogeneous (→ random) way**

- Configuration: $\mathbf{x} = (x_1, \cdots, x_D) \in \mathcal{M}_D, \quad D \gg 1$
- Dynamics: $\partial_t \, x_i(t) = f_i(\mathbf{x}(t), \hat{a}) + \eta_i(t)$ $\quad \hat{a}$ randomness $\quad \langle \eta^2(t) \rangle \propto \beta^{-1}$



Equilibria $\mathbf{x}^*$: $\partial_t \, x_i^* = f_i(\mathbf{x}^*, \hat{a}) = 0 \quad$ for all i

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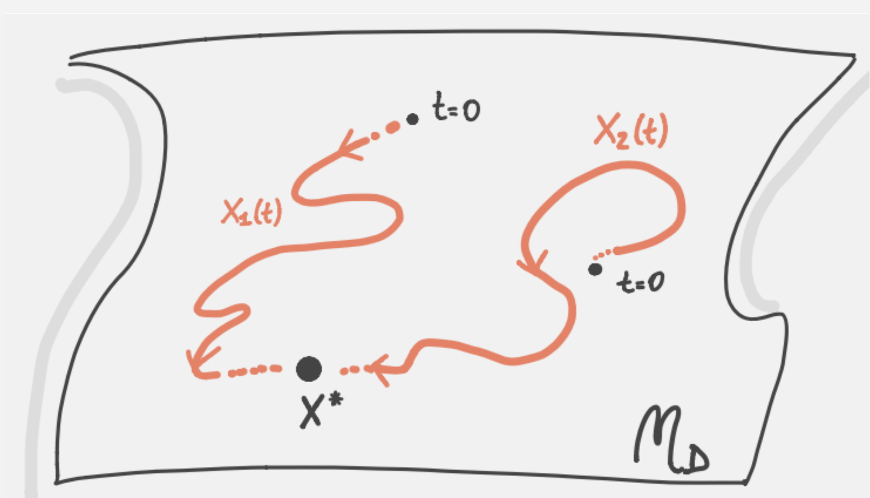

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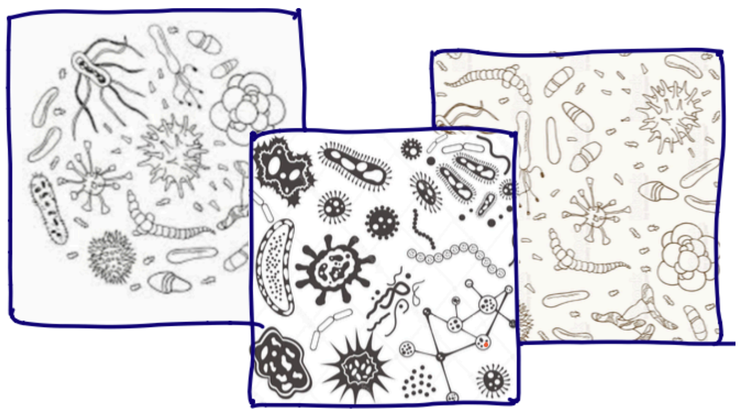
(1) High-D & eterogeneous interactions produce "glassiness": **huge number** $\mathcal{N} \sim e^{D\Sigma}$ **of competing, very different equilibria** [Σ= "complexity"]



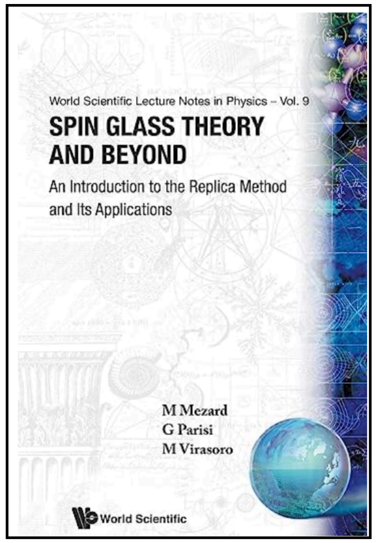
different equilibria with same diversity

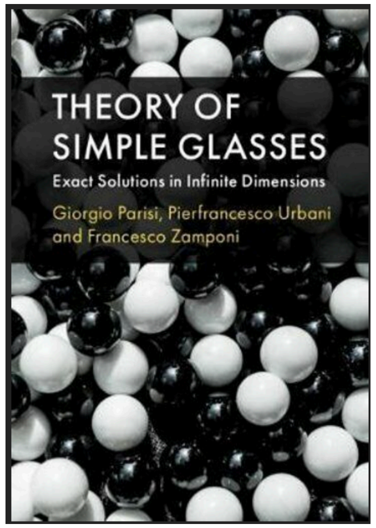
(2) Dynamics with many attractors can be complex: **slow (aging), chaotic, intermittent, with avalanches, activated**...

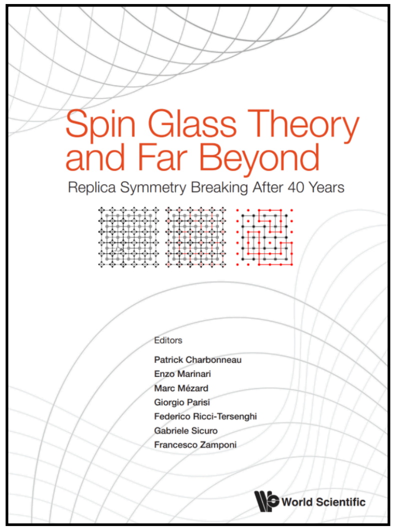
Purpose: understand this dynamics quantitatively.

Approach: count & classify all equilibria as a function of “typical” properties (e.g. stability). Statistics.

A long history in the field of glasses & spin glasses:





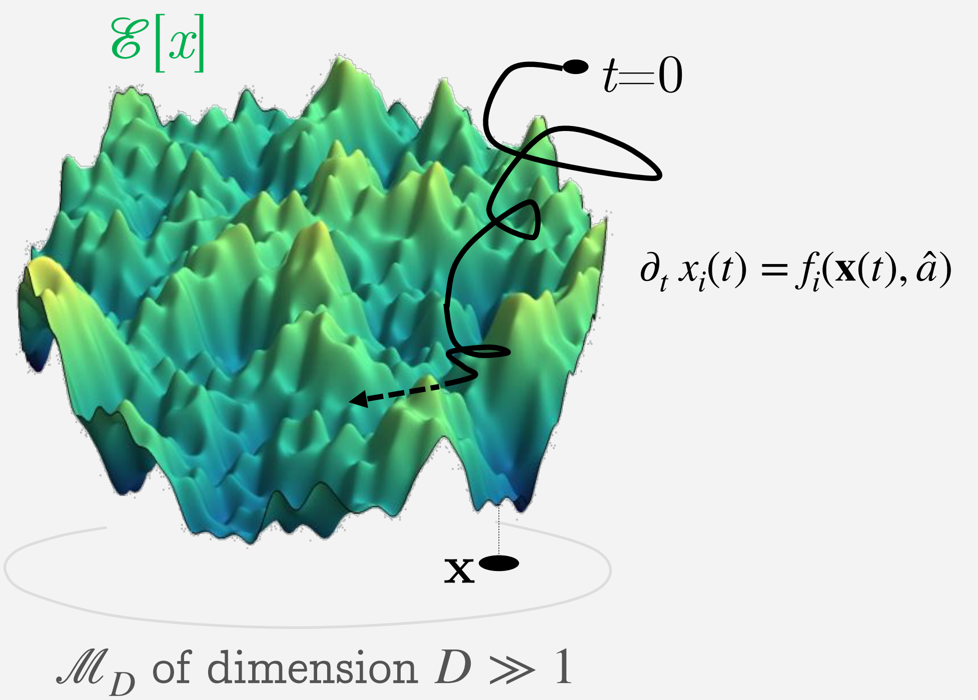


https://dissyskcl.github.io/news/rsb40/

The optimization paradigm...

Conservative problems: $\partial_t x_i = f_i(\mathbf{x}, \hat{\alpha}) = -\partial_{x_i}\mathscr{E}(\mathbf{x}, \hat{\alpha})$

Equilibria are stationary points of **high-D landscape** $\mathscr{E}(\mathbf{x}, \hat{a})$.



- **Fitness landscapes** in evolutionary biology

  Park, Hwang, Krug JPhysA 53 (2020) [....]

- **Loss landscapes** in machine (supervised) learning

  Baskerville et al JPhysA 55 (2022) [....]

- **Cost landscapes** in inference & constraint satisfaction

  Fedeli, Fyodorov JSP 175 (2019) [....]

- **Energy landscapes** in condensed/soft matter, e.g.

  $\mathbf{x}$ conf of particles/spins, $\mathscr{M}_D$ sphere, $\sum_{i=1}^{D} x_i^2 = D$

  $\mathscr{E}(\mathbf{x}, \hat{\alpha}) = \sum_{p=2}^{\infty} \sum_{i_1,\cdots,i_p} a^{(p)}_{i_1\cdots i_p} x_{i_1}\cdots x_{i_p}$, with $a^{(p)}_{i_1\cdots i_p}$ random

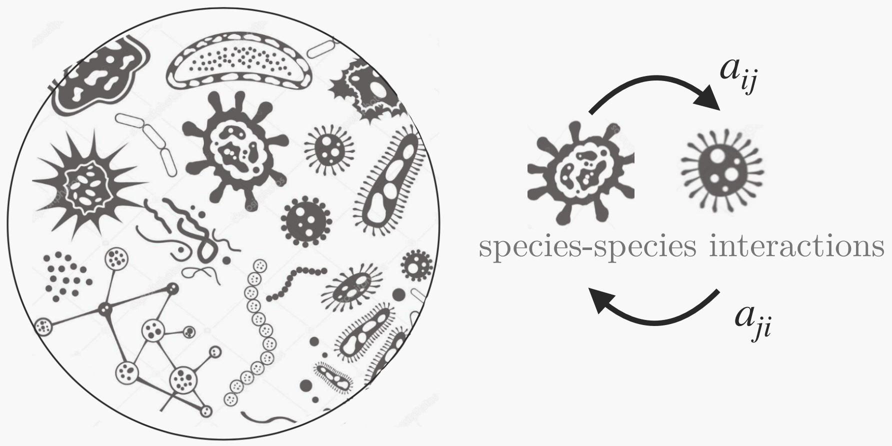
  "spherical p-spin models" ← effective model structural glasses

  Kirkpatrick, Thirumalai, Wolynes 1989

  Crisanti, Sommers 1992

...and beyond: non-reciprocity.

Non-conservative problems: $\partial_t x_i = f_i(\mathbf{x}, \hat{\alpha}) \neq \partial_i \mathscr{E}$, because of **non-reciprocal (asymmetric) interactions.**



- **Interacting neurons** in neuroscience

  Sompolinski, Crisanti, Sommers 1988 [....]

- **Interacting firms** (or traders, or banks)

  Moran, Bouchaud 2019 [....]

- **Gene-regulatory networks**

  → **A. Annibale talk**

- **Interacting species** in ecology, e.g.

  $\mathbf{x}$ species abundance, $\mathscr{M}_D = \mathbb{R}_+^D$

  $$f_i(\mathbf{x}, \hat{\alpha}) = x_i \left(\kappa_i - x_i - \sum_j a_{ij} x_j \right), \quad \text{with } a_{ij} \neq a_{ji}$$

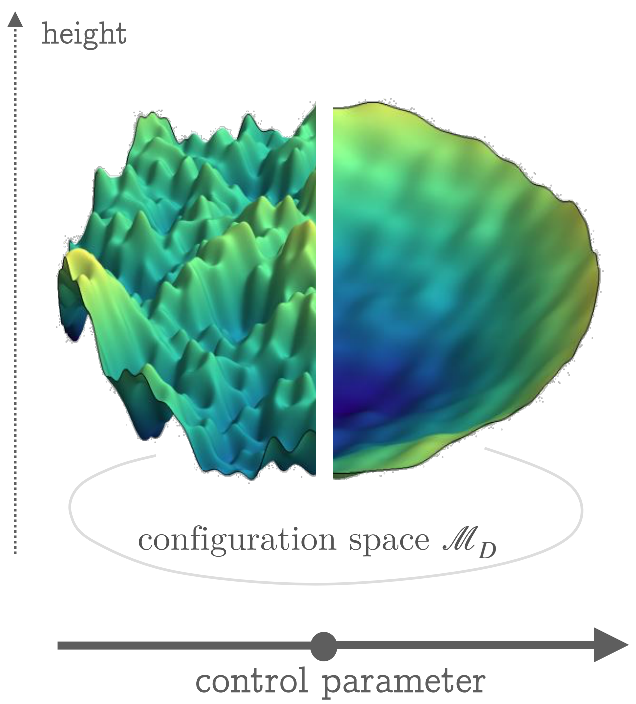
  "Generalized random Lotka-Volterra equations"

  May 1972

The program.

1. Count

Glassiness or not?
"Topology trivialization" transitions?



2. Classify

How many at a given height,
or with fixed fraction of $x_i > c$?
Which are stable or unstable?
Distribution and connectivity in configuration space?

3. Link to dynamics

Which attractors trap system at shorter times? At longer times? Most probable dynamical paths?
Aging, activated jumps, chaos?

An example & two questions

Simple Gaussian landscapes.

Configuration space $\mathscr{M}_D$: sphere $\sum_{i=1}^{D} x_i^2 = D$

Gaussian landscape $\mathscr{E}_p(\mathbf{x}) = \sum_{i_1,\cdots,i_p} a_{i_1\cdots i_p} x_{i_1} \cdots x_{i_p} \qquad p \geq 3$

Isotropic correlations: $\left\langle \mathscr{E}_p(\mathbf{x})\, \mathscr{E}_p(\mathbf{x}') \right\rangle = \frac{D}{2} \left(\frac{\mathbf{x} \cdot \mathbf{x}'}{D} \right)^p$

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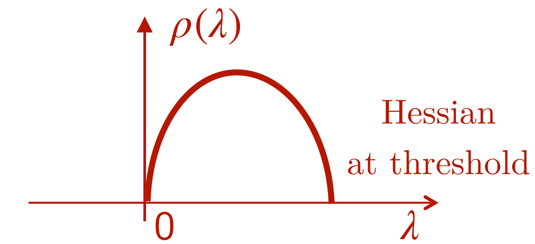

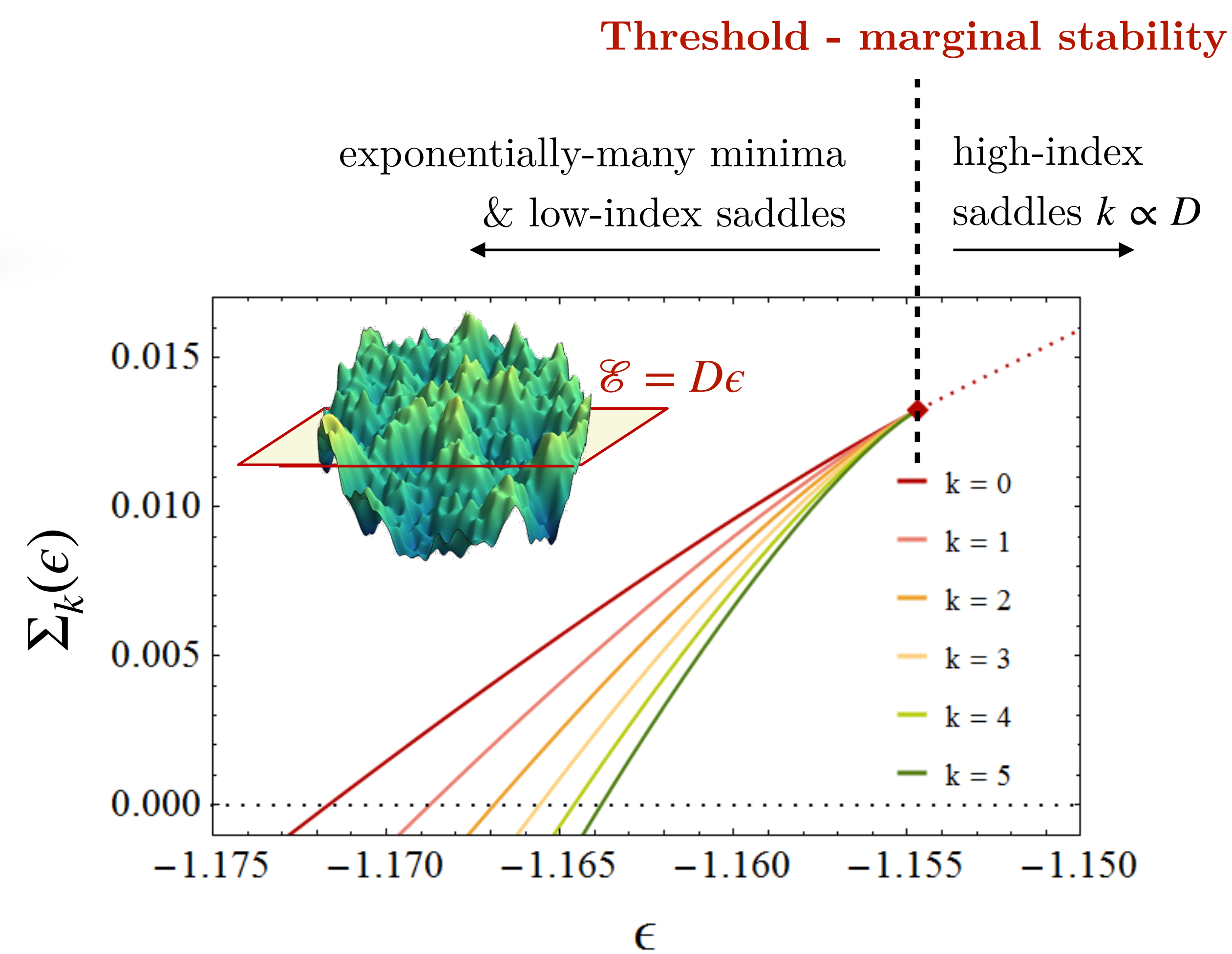

■ **Count & classify**

$\mathscr{N}_k(\epsilon)$ = number of equilibria $\mathbf{x}^*$ at $\mathscr{E} = D\,\epsilon$.

Quenched complexity $\Sigma_k(\epsilon) = \lim_{D\to\infty} \frac{\langle \log \mathscr{N}_k(\epsilon)\rangle}{D}$

Cavagna, Giardina, Parisi 1998
Auffinger, Ben Arous, Cerny 2013

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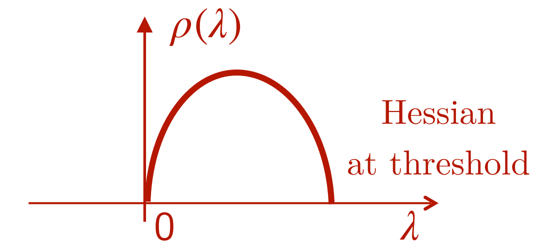

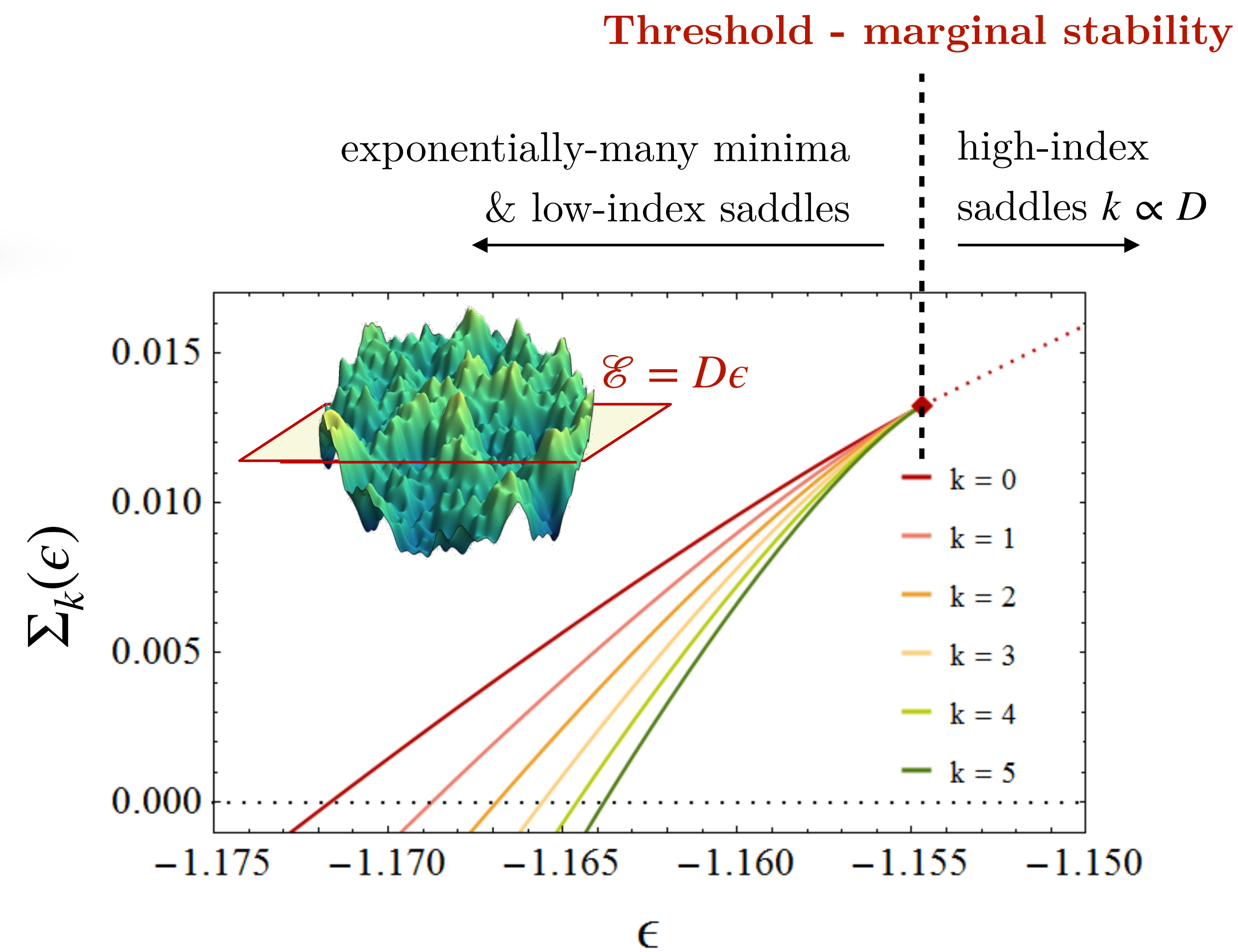

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Link to dynamics

"Short-time" dynamics $t \sim O(D^0)$ approaches asymptotically the threshold energy [marginally stable minima] and *ages*

To explore bottom of the landscape (and eventually equilibrate) need $t(D) \sim O(e^D)$: jumps between stable minima.

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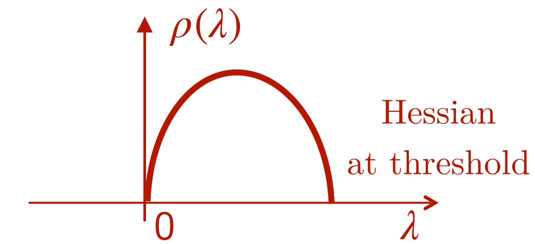

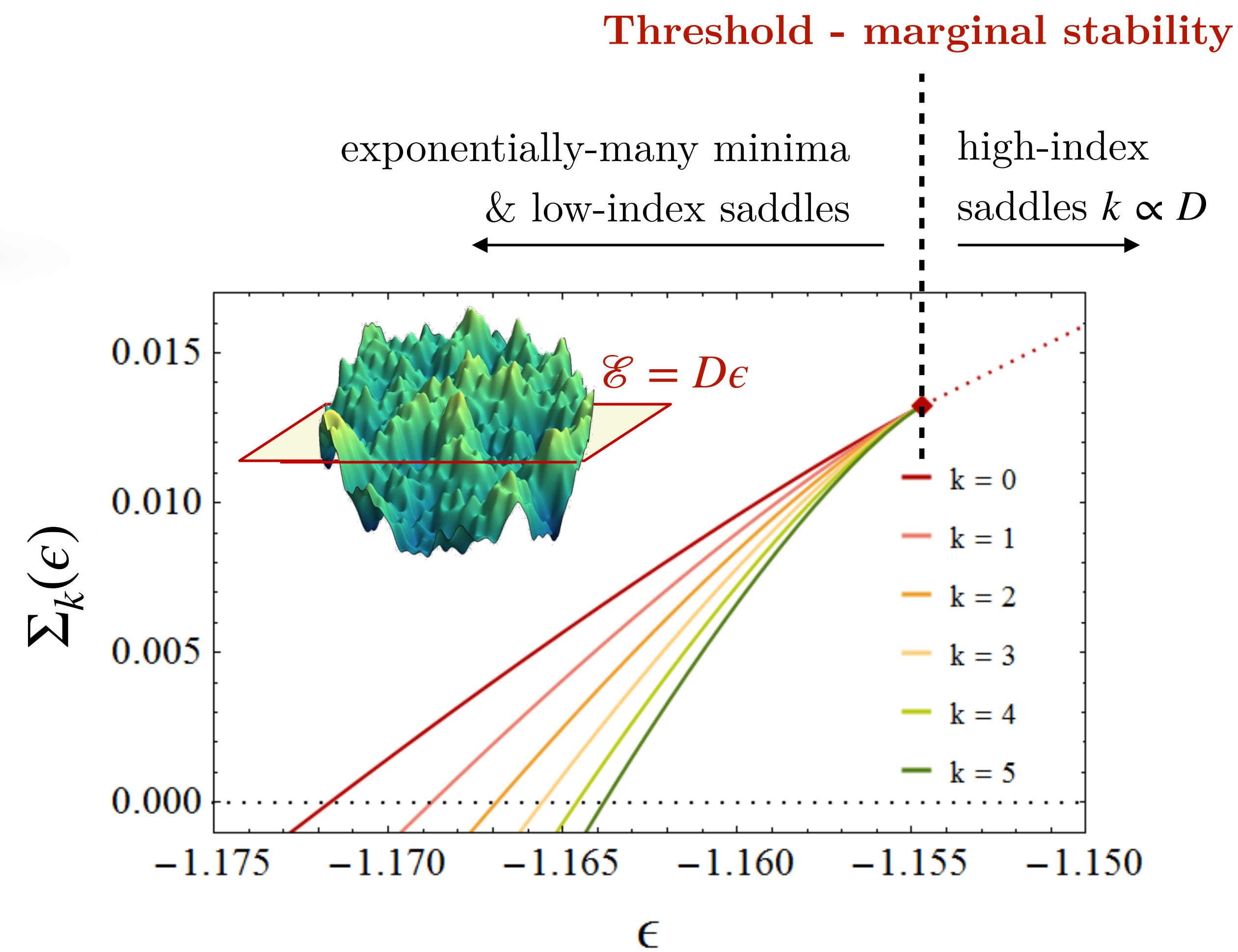

- **Count & classify**

The program

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This talk: two questions.

- **Where are unstable attractors, i.e. the saddles?**
- **What happens when the landscape picture breaks down?**

This talk: outline.

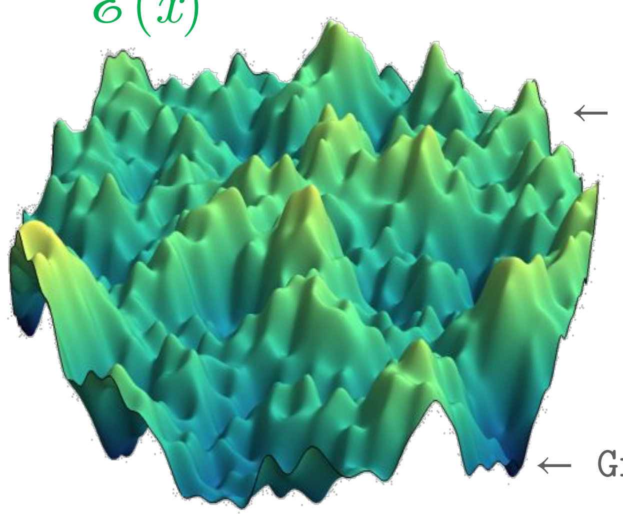
- Beyond “standard” landscape-based tools
- Direct counting, & how random matrix theory helps
- The two questions: why relevant, what we can say about them

“Standard” tools & more recent inputs

“Standard” glassy counting techniques.

Standard recipes involve **“tweaked” equilibrium calculations** [$\sim$ large deviations]

Franz and Parisi 1995
Monasson 1995



Thermodynamics: $\mathscr{Z}_\beta = \int dx e^{-\beta \mathscr{E}(x)}$

When $\beta \to \infty$, selects deepest minima (GS).

How to pick up & count the $\mathscr{N} \sim e^{D\Sigma}$ local minima (metastable states)?

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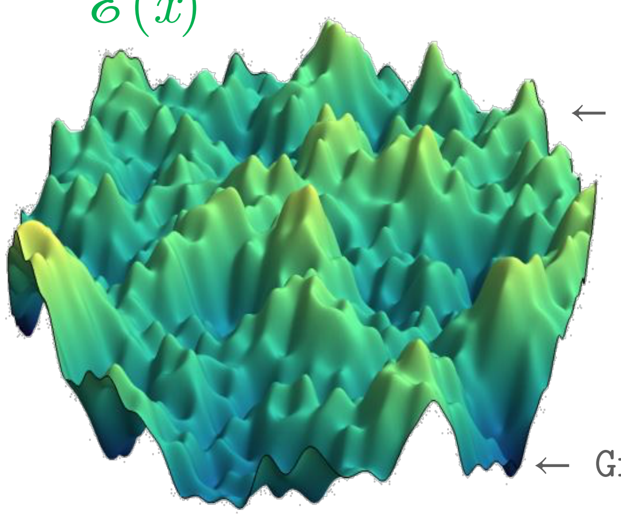

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The Monasson method

Free energy of m weakly coupled "real replicas"

$$F(m,\beta) = \lim_{D\to\infty,\epsilon\to 0} D^{-1} \left\langle \log \left[\int \prod_{k=1}^{m} dx^{(i)} e^{-\beta \sum_k \mathscr{E}(x^{(k)}) + \epsilon \sum_{kl} x^{(k)} x^{(l)}} \right] \right\rangle$$

Related to number of metastable states by Legandre transform: $F(m,\beta) \sim fm - \beta^{-1} D^{-1} \log \mathscr{N}(f,\beta)$

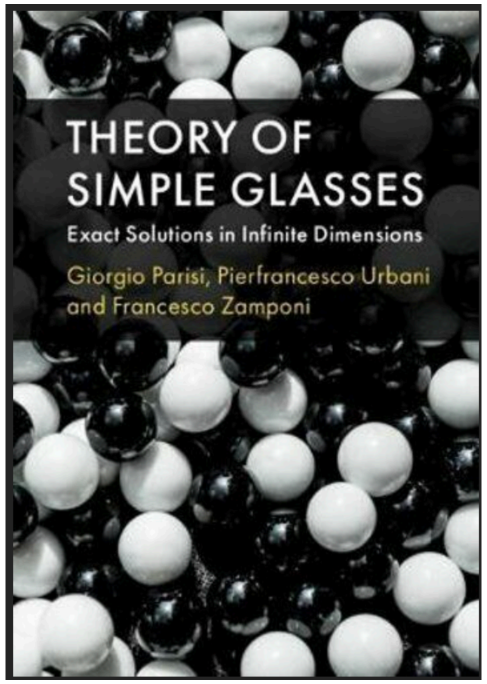
Reconstruct parametrically the complexity Σ:

$$f = \partial_m F(m,\beta) \qquad \Sigma = D^{-1} \log \mathscr{N} = m^2 \partial_m \left(\frac{\beta F(m,\beta)}{m} \right)$$

(Take $\beta \to \infty$ at the end: free energy $f \to$ energy ϵ)

Developments: Mueller, Leuzzi, Crisanti 2006

“Standard” glassy counting techniques very insightfull.



however:

1. Need a potential function/ energy landscape.

2. Pick up stable (marginally) stationary points, i.e. local minima.

Another approach: Kac-Rice formula(s).

Number $\mathcal{N}(\phi)$ of equilibria $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = \left(-\nabla \mathcal{E}(\mathbf{x}^*)\right) = 0$ and $\Phi(\mathbf{x}^*) = \phi$ (arbitrary constraints)

Random variable with scaling: $\mathcal{N}(\phi) \sim e^{D\,\Sigma(\phi)+o(D)}$.

"Kac-Rice formula": recipe to compute moments of $\mathcal{N}(\phi)$

$$\mathbb{E}[\mathcal{N}(\phi)] = \int_{\mathcal{M}_D} d\mathbf{x}\,\mathcal{P}_{\mathbf{x}}\left(\mathbf{f}=\mathbf{0}\right)\mathbb{E}_{\mathbf{x}}\left[\left|\det\left(\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right)\right|\chi_{\Phi(\mathbf{x})=\phi}\ \Big\|\ \mathbf{f}=\mathbf{0}\right]$$

Higher moments:

$$\mathbb{E}[\mathcal{N}^n(\phi)] = \int_{\mathcal{M}_D^{\otimes n}} \prod_{m=1}^{n} d\mathbf{x}^{(m)}\,\mathcal{P}_{\{\mathbf{x}^{(m)}\}}\left(\left\{\mathbf{f}^{(m)}=\mathbf{0}\right\}\right)\mathbb{E}_{\{\mathbf{x}^{(m)}\}}\left[\prod_{m=1}^{n}\left|\det\left(\frac{\partial f_i(\mathbf{x}^{(m)})}{\partial x_j^{(m)}}\right)\right|\chi_{\Phi(\mathbf{x}^{(m)})=\phi}\ \Big\|\ \left\{\mathbf{f}^{(m)}=\mathbf{0}\right\}\right]$$

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Complexity via replica trick: $\Sigma(\phi) = \lim_{D\to\infty} \dfrac{\mathbb{E}[\log \mathcal{N}(\phi)]}{D} = \lim_{D\to\infty}\lim_{n\to 0} \dfrac{\mathbb{E}[\mathcal{N}^n]-1}{Dn}$

The recent input: Random Matrix Theory tooolbox.

$$\mathbb{E}[\mathcal{N}^n(\phi)] = \int_{\mathcal{M}_D^{\otimes n}} \prod_{m=1}^{n} d\mathbf{x}^{(m)} \, \mathcal{P}_{\{\mathbf{x}^{(m)}\}}\left(\{\mathbf{f}^{(m)} = \mathbf{0}\}\right) \mathbb{E}_{\{\mathbf{x}^{(m)}\}}\left[\prod_{m=1}^{n} \left|\det\left(\frac{\partial f_i(\mathbf{x}^{(m)})}{\partial x_j^{(m)}}\right)\right| \chi_{\Phi(\mathbf{x}^{(m)})=\phi} \;\Big\|\; \{\mathbf{f}^{(m)} = \mathbf{0}\}\right] \sim e^{D\Sigma^{(n)} + o(D)}$$

- Since forces $f_i(\mathbf{x})$ are random, need to control **random matrix field** $\hat{M}_{ij}[\mathbf{x}] = \frac{\partial f_i(\mathbf{x})}{\partial x_j}$ Fyodorov (2004)
- Problem of coupled, conditioned random matrices
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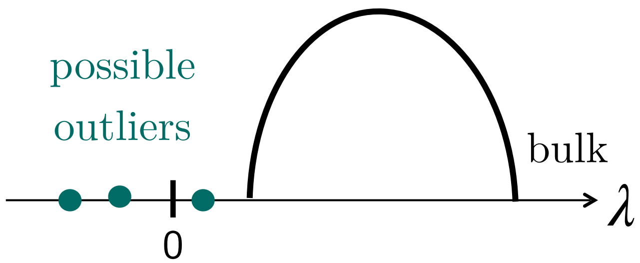
Stability of equilibria is encoded in the spectrum of matrices.
For conservative problems, this is Hessian field $\hat{M}_{ij}[\mathbf{x}] = \partial^2_{x_i x_j}\mathcal{E}(\mathbf{x})$

Typical spectrum of the Hessians:



With replicas, though.

Exponentially-large random quantities $\mathcal{N} \sim e^{D\,\Sigma_D + o(D)}$ are typically **not self-averaging.**

$$\Sigma^{(A)} = \lim_{D\to\infty} \frac{\log \mathbb{E}[\mathcal{N}]}{D} \qquad \Sigma^{(Q)} = \lim_{D\to\infty} \frac{\mathbb{E}[\log \mathcal{N}]}{D} = \lim_{D\to\infty} \lim_{n\to 0} \frac{\mathbb{E}[\mathcal{N}^n] - 1}{Dn}$$

By convexity: $\Sigma^{(A)} \geq \Sigma^{(Q)}$

Results (even rigorous) on annealed complexity via Kac-Rice:

Fyodorov 2005-2021
Ben Arous & Auffinger, 2011-2021
Auffinger, Ben Arous, Černý 2013 – Wainrib & Touboul 2013 – Fyodorov & Khoruzhenko 2016 – Ge & Ma 2017
Ipsen & Forrester 2018 – Ben Arous, Mei, Montanari & Nica 2019 – Maillard, Ben Arous, Biroli 2020
Ben Arous, Fyodorov, Khoruzhenko 2020
Lacroix-A-Chez-Toine & Fyodorov 2022
Lacroix-A-Chez-Toine, Fyodorov, Fedeli 2023
[...]

"Replicated Kac-Rice" for quenched complexity.

Three ingredients: **isotropy** (rotational symmetry), **Gaussianity**, **concentration** (of Hessian, e.g. $\rho_D(\lambda)$)

VR, Ben Arous, Biroli, Cammarota – Physical Review X 9 (2019)

Gaussianity, Isotropy, Concentration.

$$\mathbb{E}[\mathcal{N}^n(\phi)] = \int_{\mathcal{M}_D^{\otimes n}} \prod_{a=1}^{n} d\mathbf{x}^{(a)}\, \mathcal{P}_{\{\mathbf{x}^{(a)}\}}\left(\{\mathbf{f}^{(a)}=\mathbf{0}\}\right) \mathbb{E}_{\{\mathbf{x}^{(a)}\}}\left[\prod_{a=1}^{n}\left|\det\left(\frac{\partial f_i(\mathbf{x}^{(a)})}{\partial x_j^{(a)}}\right)\right|\chi_{\Phi(\mathbf{x}^{(a)})=\phi}\;\Big\|\;\{\mathbf{f}^{(a)}=\mathbf{0}\}\right] \sim e^{D\Sigma^{(n)}+o(D)}$$

Gaussianity

- All determined by covariances, can be computed explicitly

$$C^{ab}_{ij,kl} = \left\langle \frac{\partial f_i(\mathbf{x}^{(a)})}{\partial x_j^{(a)}} \frac{\partial f_k(\mathbf{x}^{(b)})}{\partial x_l^{(b)}} \right\rangle_c$$

- Can treat explicitly conditioning to $\mathbf{f}^{(a)} = \mathbf{0}, \Phi(\mathbf{x}^{(a)}) = \phi$

→ finite rank perturbations

Isotropy

- Joint distributions depend only on order parameters

$$Q^{ab} = D^{-1}\,\mathbf{x}^{(a)} \cdot \mathbf{x}^{(b)}, \quad m^a = D^{-1}\,\mathbf{x}^{(a)} \cdot \mathbf{1}$$

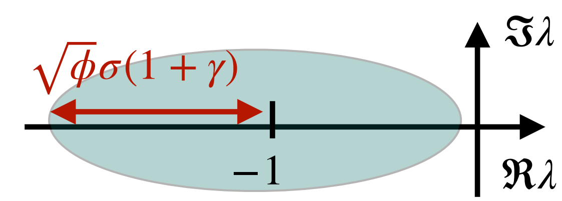
$$\int_{\mathcal{M}_D^{\otimes n}} \prod_{a=1}^{n} d\mathbf{x}^{(a)} \to \int \prod_{a,b=1}^{n} dQ^{ab}$$

- Invariant statistics of random matrices: GOE, elliptic,...

Concentration

- Low-rank perturbations do not affect $\rho(\lambda)$at leading order

supp[$\rho(\lambda)$] — evalue density of M

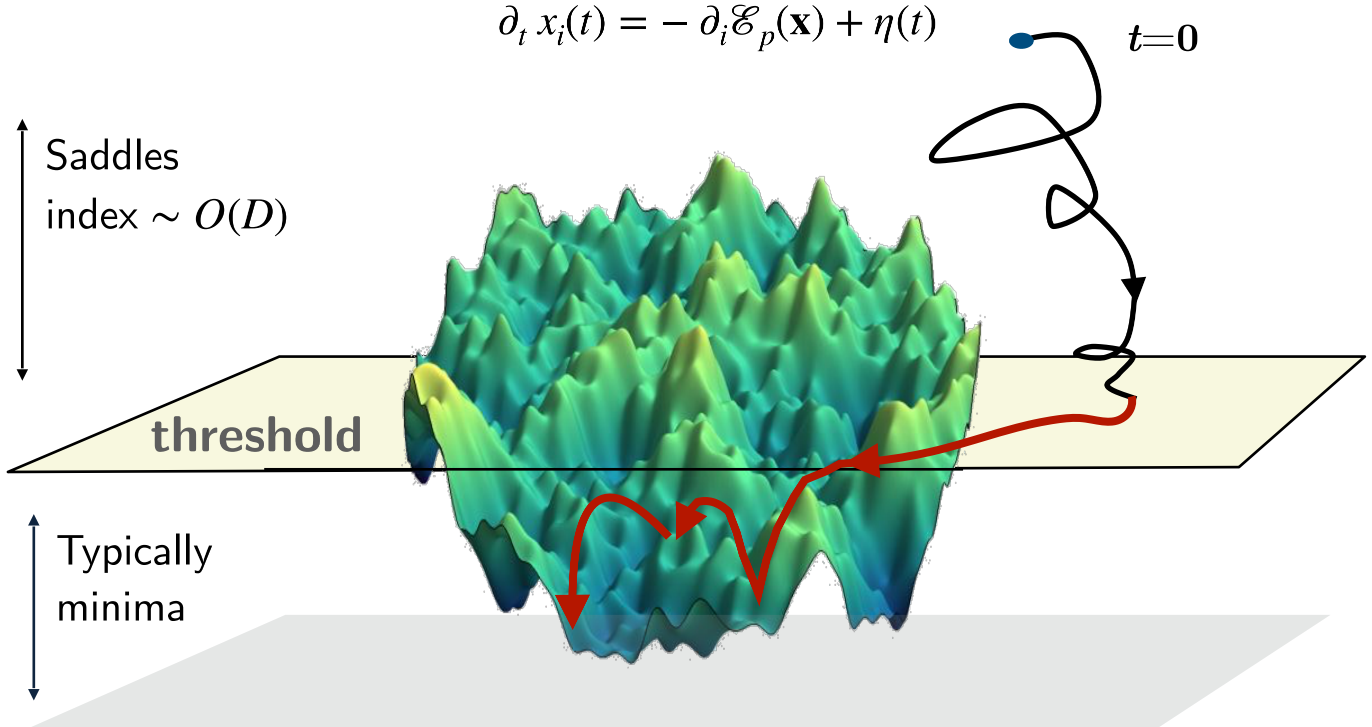


- Variational problem: self-consistent equations for Q^{ab}, m^a .

For more details: VR, Ben Arous, Biroli, Cammarota, Physical Review X 9 (2019)

Question 1: where are the unstable attractors?

Motivation: activated dynamics.



Descent + aging at threshold

$t \gg 1$ but $t \neq O(D)$

Described by mean-field dynamical equations

Activated dynamics: jumps between minima, crossing barriers

$\tau_{\text{jump}} \sim O(e^{D\Delta\epsilon})$, $\Delta\epsilon$ = energy barrier

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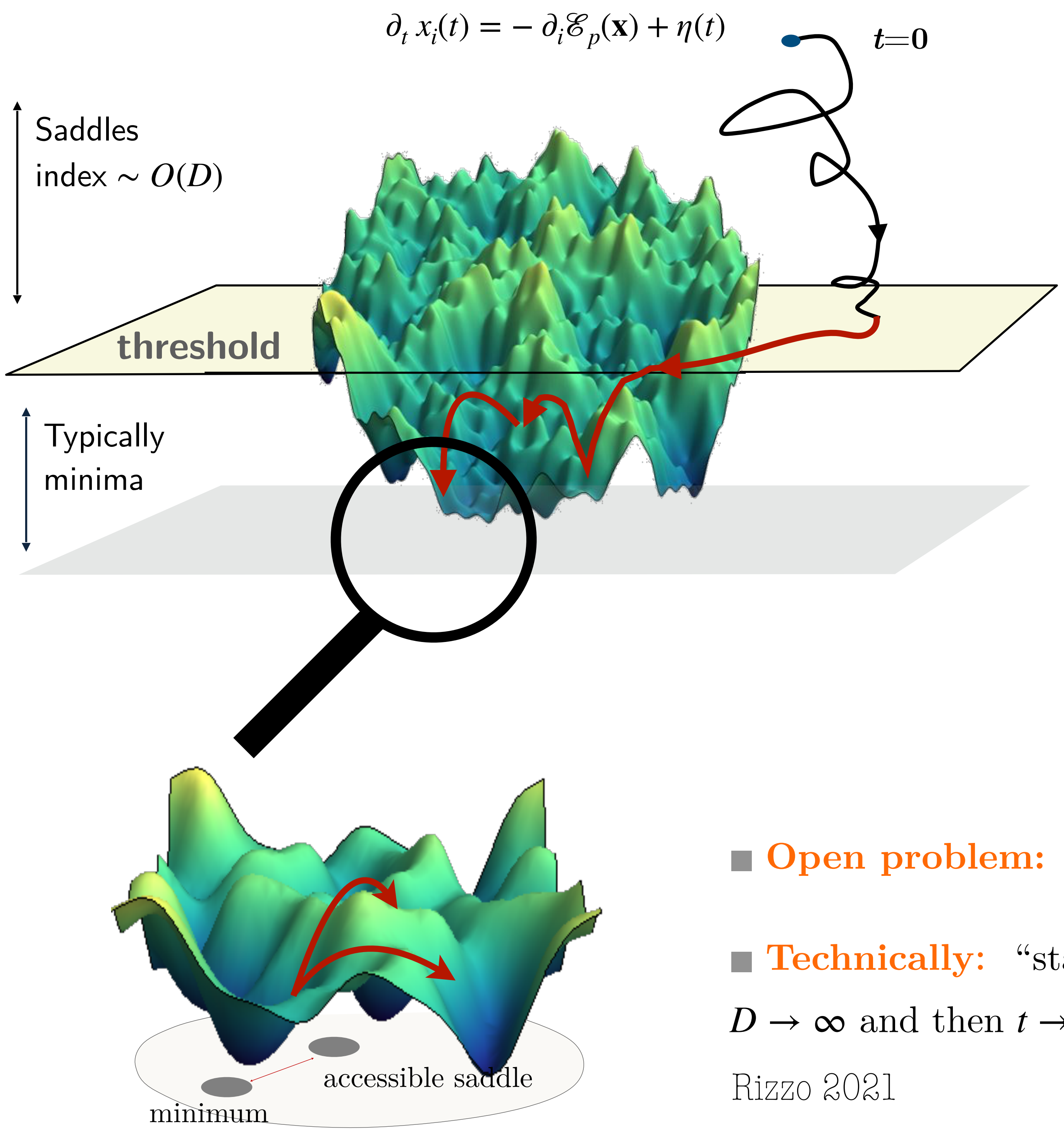

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Described by mean-field dynamical equations

Activated dynamics: jumps between minima, crossing barriers

$\tau_{\text{jump}} \sim O(e^{D\Delta\epsilon})$, $\Delta\epsilon$ = energy barrier

- **Open problem:** dynamical theory for activated regime

- **Technically:** "standard" mean-field description obtained taking $D \to \infty$ and then $t \to \infty$. Here we need $t \sim e^{D\tau} \to \infty$: HARD!

Rizzo 2021

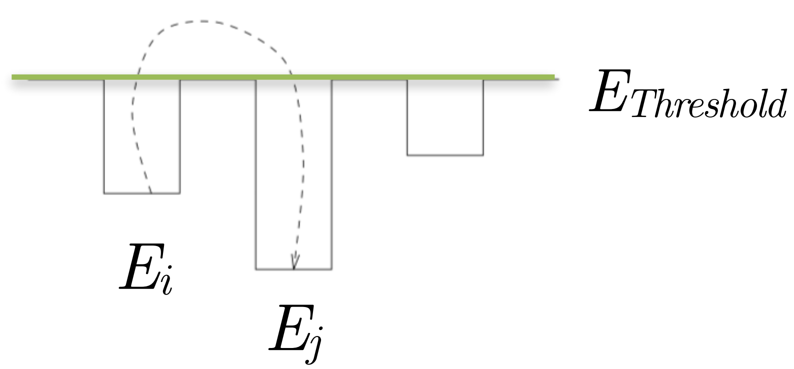
Trap models & beyond.

The Trap model paradigm

Random walk between $e^{\alpha N}$ traps of random depth via climbing up to fixed level $E_{Threshold}$

Bouchaud 1992

Dyre 1987



- Transition prob. $P(E_i \to E_j) \propto e^{-\beta(E_{Th}-E_i)}$
- Fully connected & renewal

Captures long-time dynamics (Metropolis) of **Random Energy Model** — no correlations in the landscape

Gayrard 2017

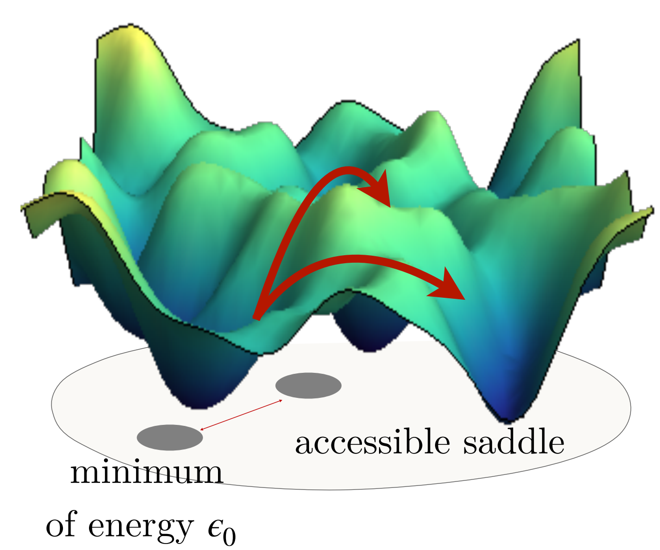
- **Dynamical approach.** Beyond fully-connected: trap model on random networks.

Margiotta, Kuhn, Sollich 2019
Tapias, Paprotzki, Sollich 2023

→ **P. Sollich talk**

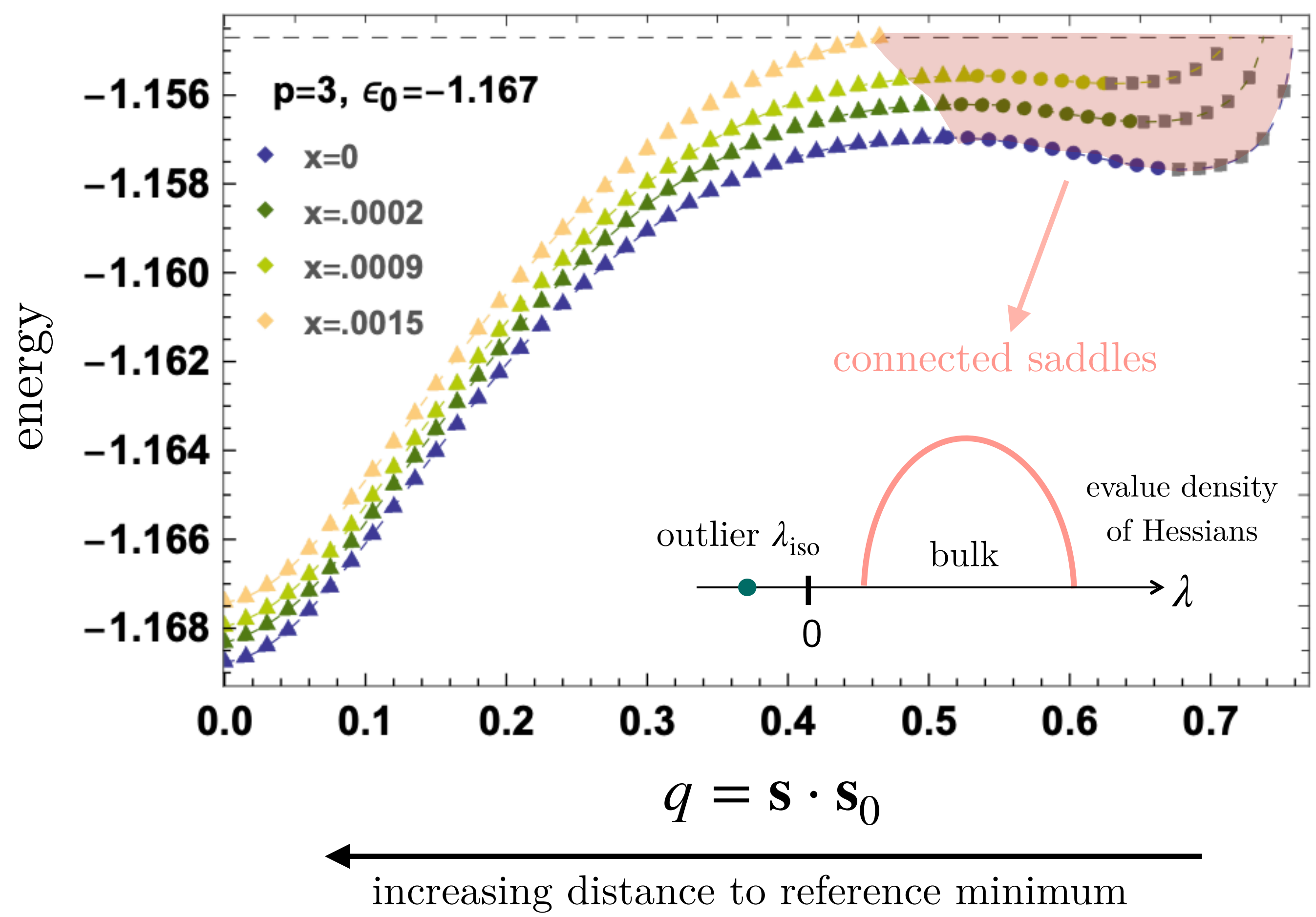
- **Landscape approach.** p-spin: a landscape with statistical correlations. Which saddles can be used to escape from one particular minimum?

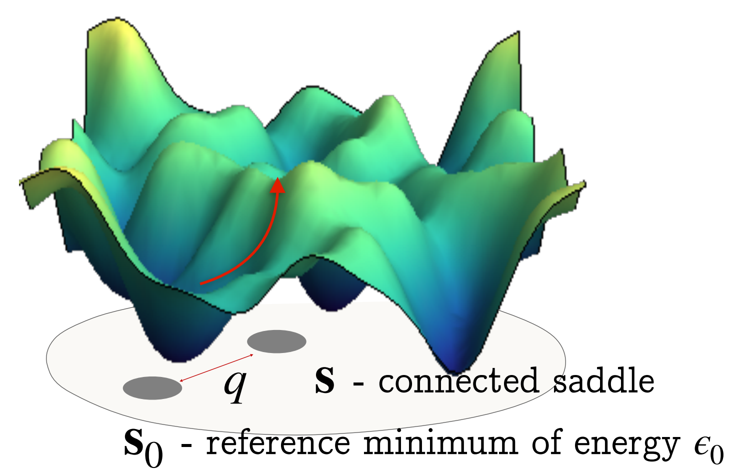
  - Barriers: how high system needs to climb up $\tau \sim e^{N\Delta\epsilon}$
  - Connectivity: which part of conf. space accessible afterwards
  - Dependence on energy of departing trap ϵ_0?



The distribution of energy barriers.

- Doubly-constrained complexity $\Sigma_1(\epsilon; q, \epsilon_0) = \lim_{D\to\infty} \frac{\langle \log \mathcal{N}_{k=1}(\epsilon; q, \epsilon_0)\rangle}{D}$: index-1 saddles, in given region





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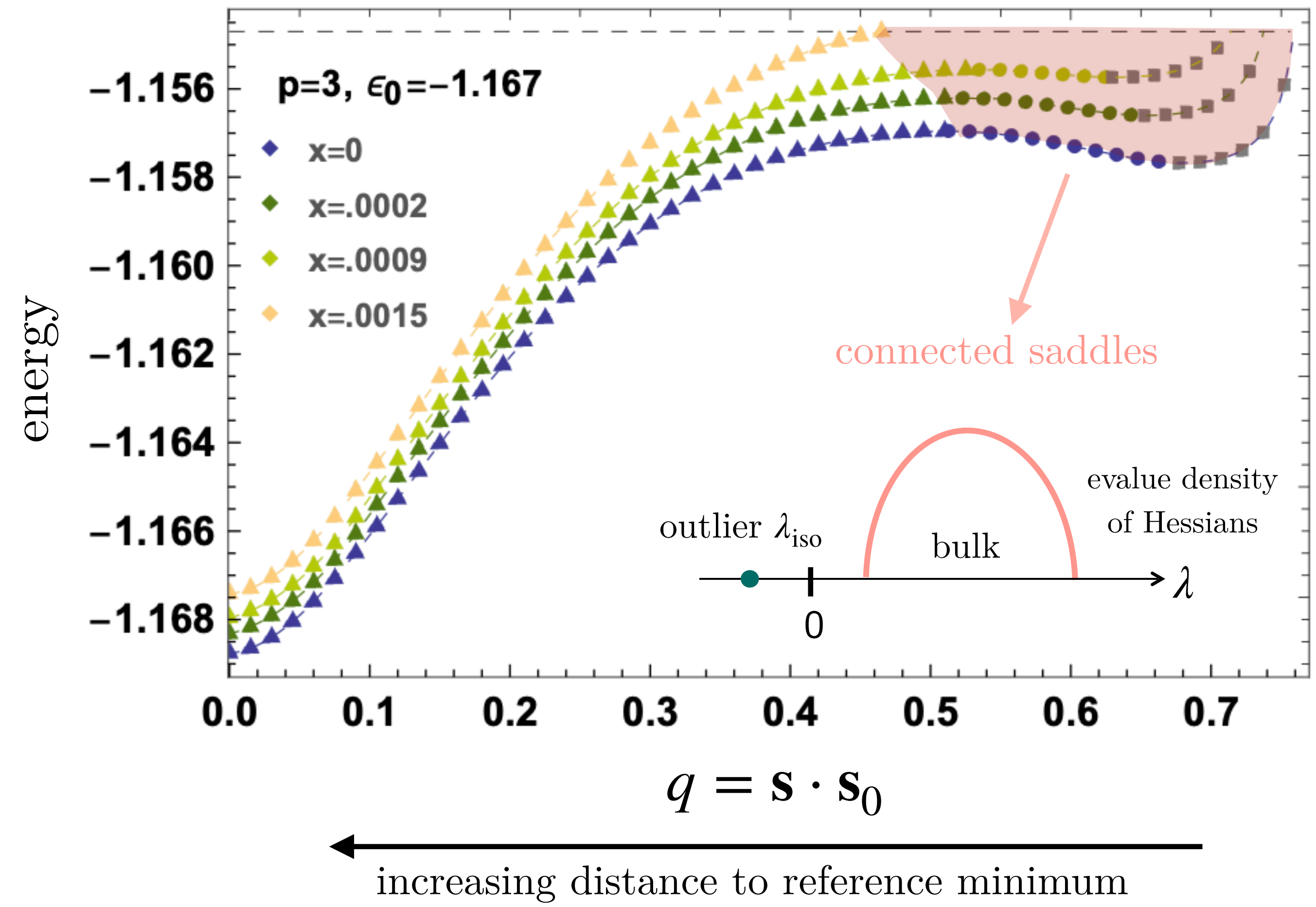
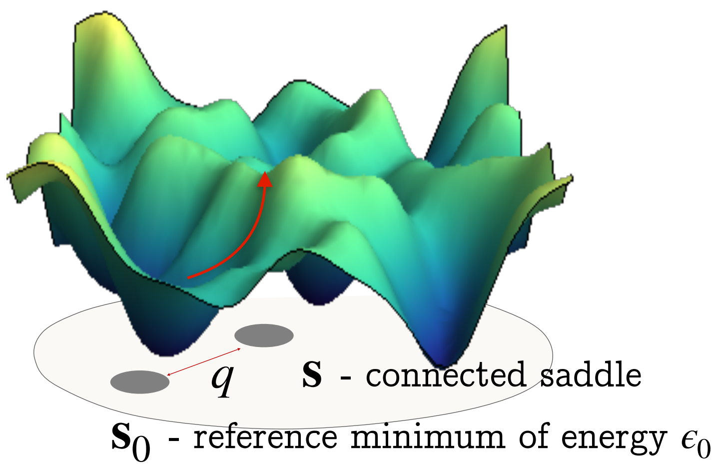


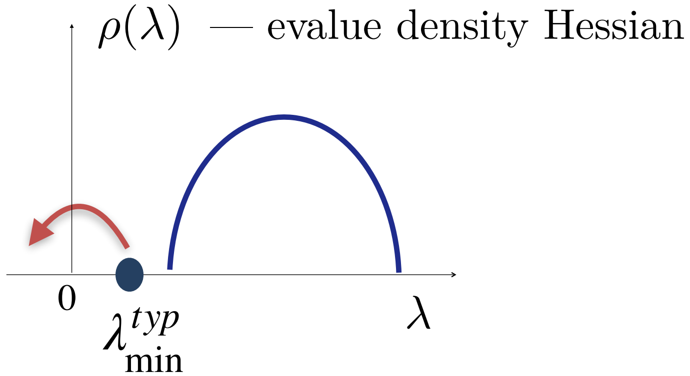


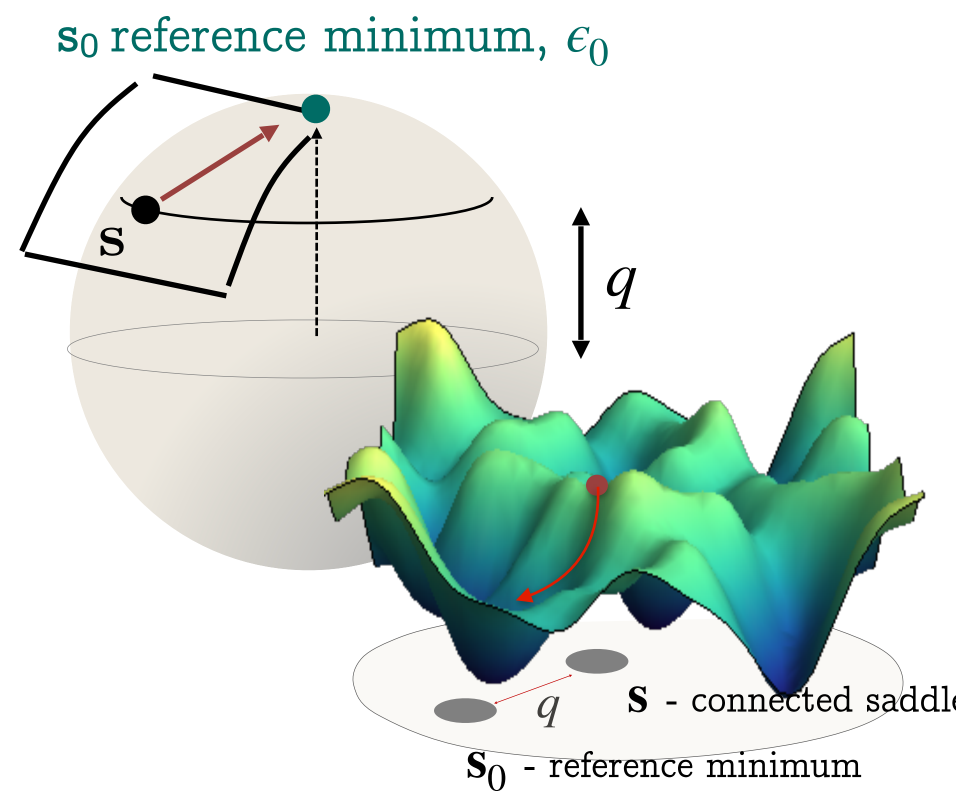
- Give access to **statistics of energy barriers → distribution of escape times** in activated dynamics
  - Optimal barrier $\Delta E = D(\epsilon^* - \epsilon_0)$ is non-linear in ϵ_0 — unlike Bouchaud trap-model
  - Deepest minima have larger convex surrounding $1 - q^*(\epsilon_0)$

For details: VR, Biroli, Cammarota EPL 126 (2), 20003 (2019) VR, J Phys A: Math Theor 53 2020

Underlying RM problem: large deviations of top eigenpair.

■ Issue: saddles are subleading: $\Sigma_{\text{saddles}} < \Sigma_{\text{minima}}$.
When targeting & counting saddles, need to condition explicitly on **unstable modes of Hessian.**





■ **Joint large deviations** of smallest Hessian eigenvalue & projection of eigenvector $\mathbf{u}$ in direction $\hat{\mathbf{e}}$ of reference minimum $u = |\mathbf{u} \cdot \hat{\mathbf{e}}|$

$$\mathbb{P}(\lambda_{min} = \lambda, u_{\text{min}} = u) = e^{-DG(\lambda,u)+o(D)}$$

$$\begin{pmatrix} & & & m^a_{1\,N-1} \\ & \mathbf{B}^a & & \vdots \\ & & & m^a_{N-2\,N-1} \\ m^a_{1\,N-1} & \cdots & m^a_{N-2\,N-1} & m^a_{N-1\,N-1} + \mu_a \end{pmatrix}$$

For details: VR, J Phys A: Math Theor 53 2020

Question 2: when there is no landscape?

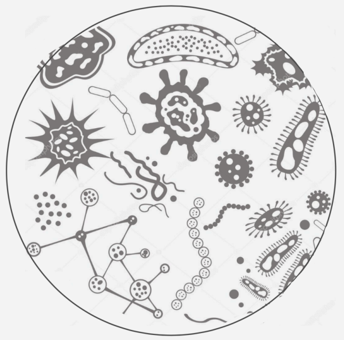
p−spin with non conservative forces: Cugliandolo, Kurchan, Le Doussal, Peliti 1997

Motivation: dynamics of complex ecosystems.

rGLVE - random Generalized Lotka-Volterra equations

$x_i(t)$ = abundance of species $i = 1, \cdots, D$

$$\frac{dx_i(t)}{dt} = x_i(t)\left(\kappa_i - x_i(t) - \sum_{j=1}^{D} \alpha_{ij} x_j(t)\right)$$



Fyodorov, Khoruzhenko 2016 Bunin 2017 Galla 2018

- Carrying capacity κ_i ($\equiv \kappa = 1$)
- Self-regulation (quadratic term)
- Random pairwise interactions

$$\langle \alpha_{ij} \rangle = \frac{\mu}{D} \qquad \text{Var}\left(\alpha_{ij}\alpha_{kl}\right) = \frac{\sigma^2}{D}\left(\delta_{ik}\delta_{jl} + \gamma\, \delta_{il}\delta_{jk}\right)$$

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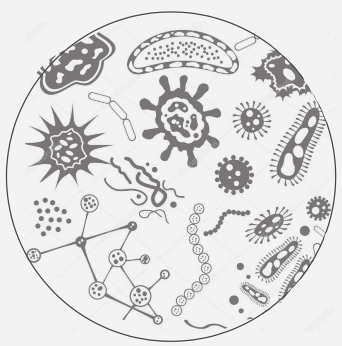

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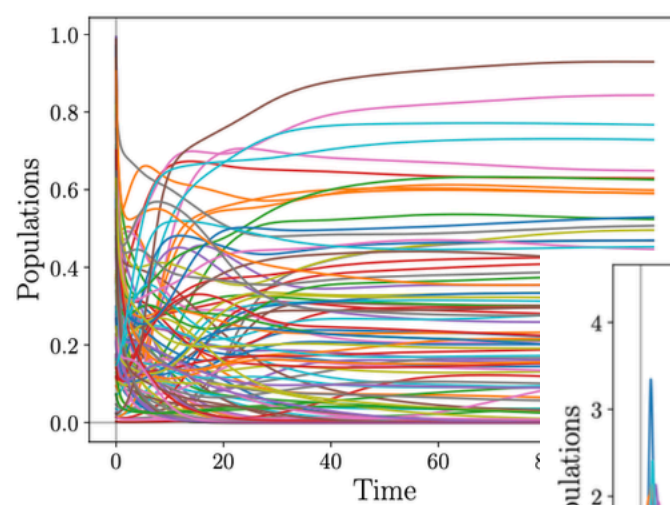
Multiple equilibria for $\sigma > \sigma_c = \frac{\sqrt{2}}{1+\gamma}$. Rieger 1989

■ Symmetric interactions ($\gamma = 1$) is a spin glass model: dynamics approaches *marginally stable minima*

Biroli, Bunin, Cammarota 2018

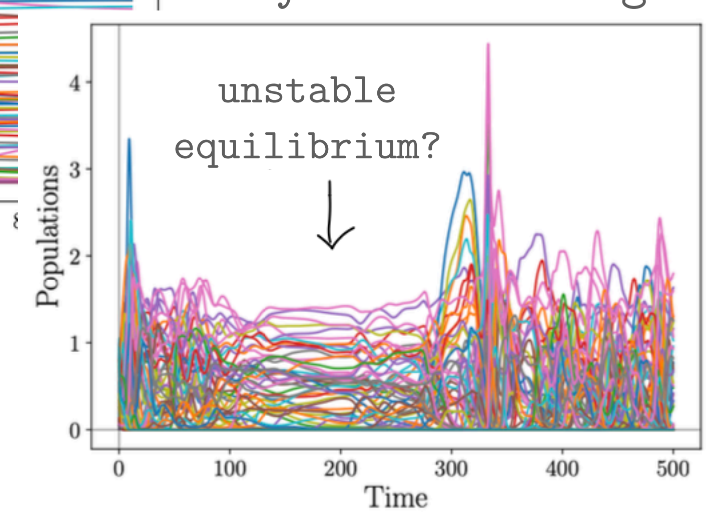
■ Asymmetric interactions ($\gamma < 1$) : properties of equilibria? Which attract dynamics, if any? Arnoulx de Pirey, Bunin 2023

Dynamics σ small



Simulations by F. Roy, 2019

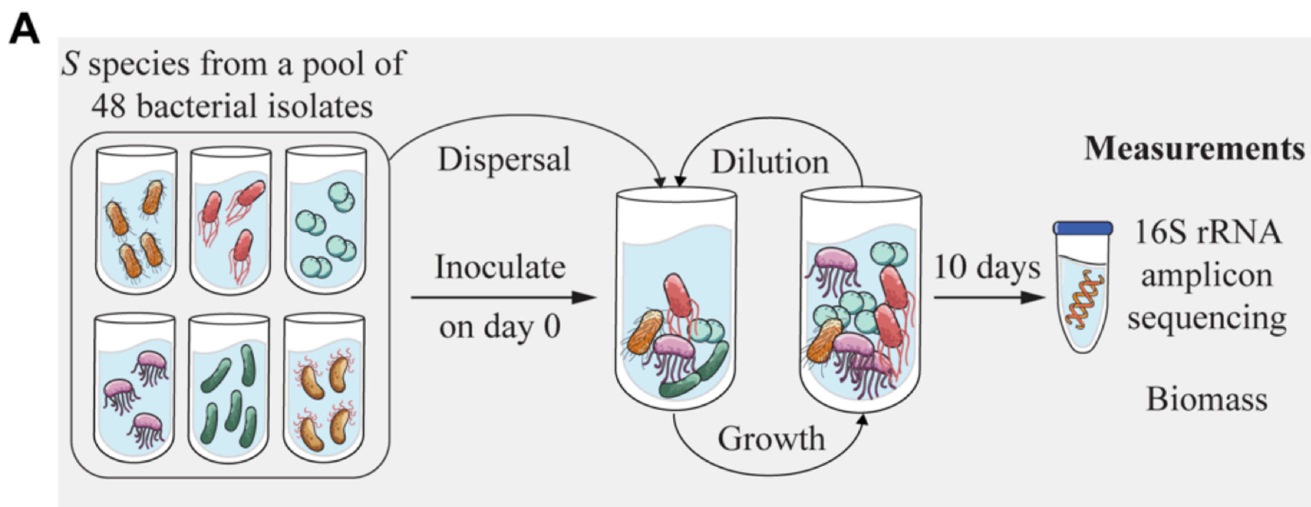
Dynamics σ large

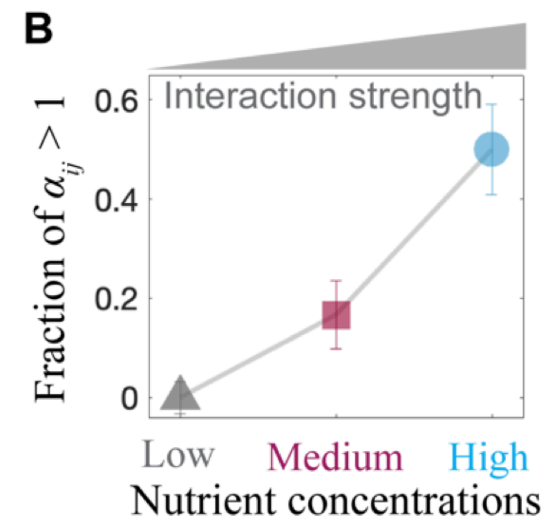


Well-mixed ecosystems in the lab.

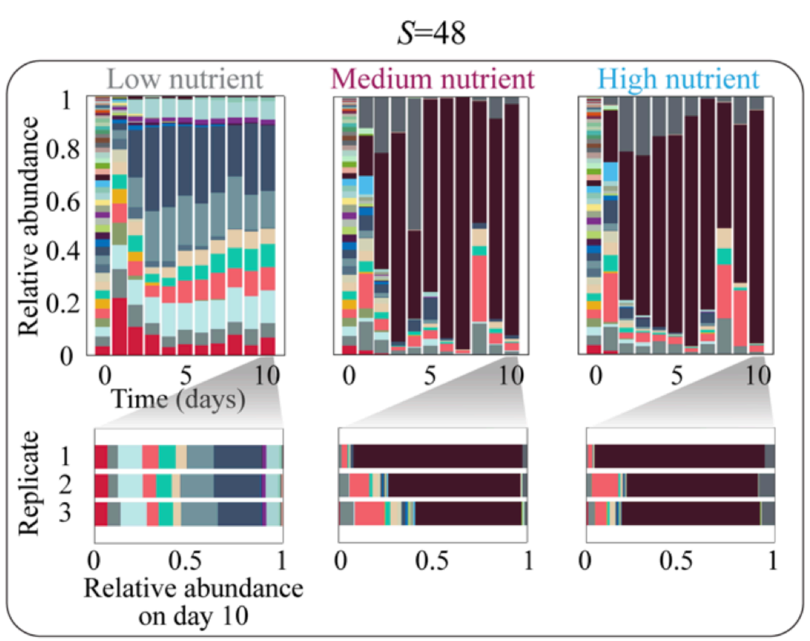
Emergent phases of ecological diversity and dynamics mapped in microcosms

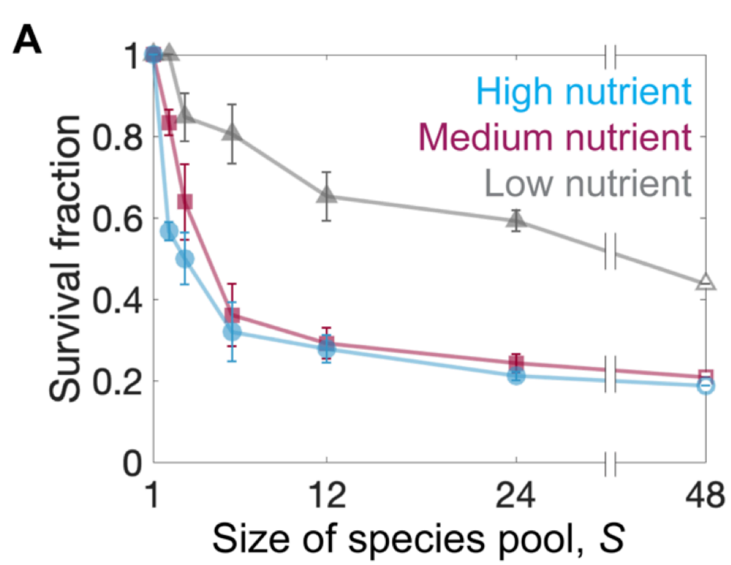
JILIANG HU, DANIEL R. AMOR, MATTHIEU BARBIER, GUY BUNIN, AND JEFF GORE Authors Info & Affiliations





Knobs: community size S and average strength μ





→ Diversity ϕ

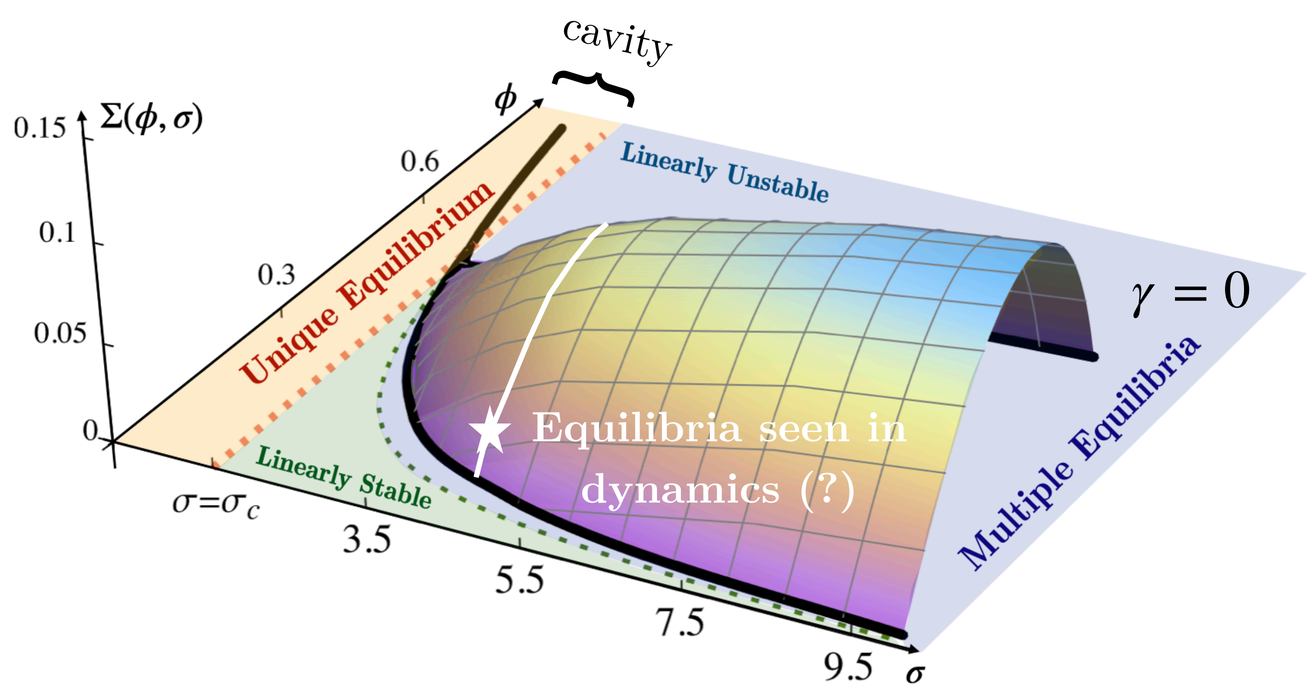
Multiple equilibria phase: diversities, (in)stability. Chaos?

- $\Sigma(\phi,\sigma) = \lim_{D\to\infty} \frac{\langle \log \mathcal{N}(\phi,\sigma)\rangle}{D}$ complexity of equilibria at fixed diversity $\phi = \frac{1}{D}\sum_{i=1}^{D} 1_{x_i^*>0}$
- Give **range of diversity accessible for dynamics** ← not fixed by marginality as for $\gamma = 1$
- All equilibria are **unstable: no marginality → chaotic dynamics, positive Lyapunov?**

Sompolinsky, Crisanti, Sommers 1988

Wainrib, Toboul 2013

Blumenthal, Rocks, Mehta 2023

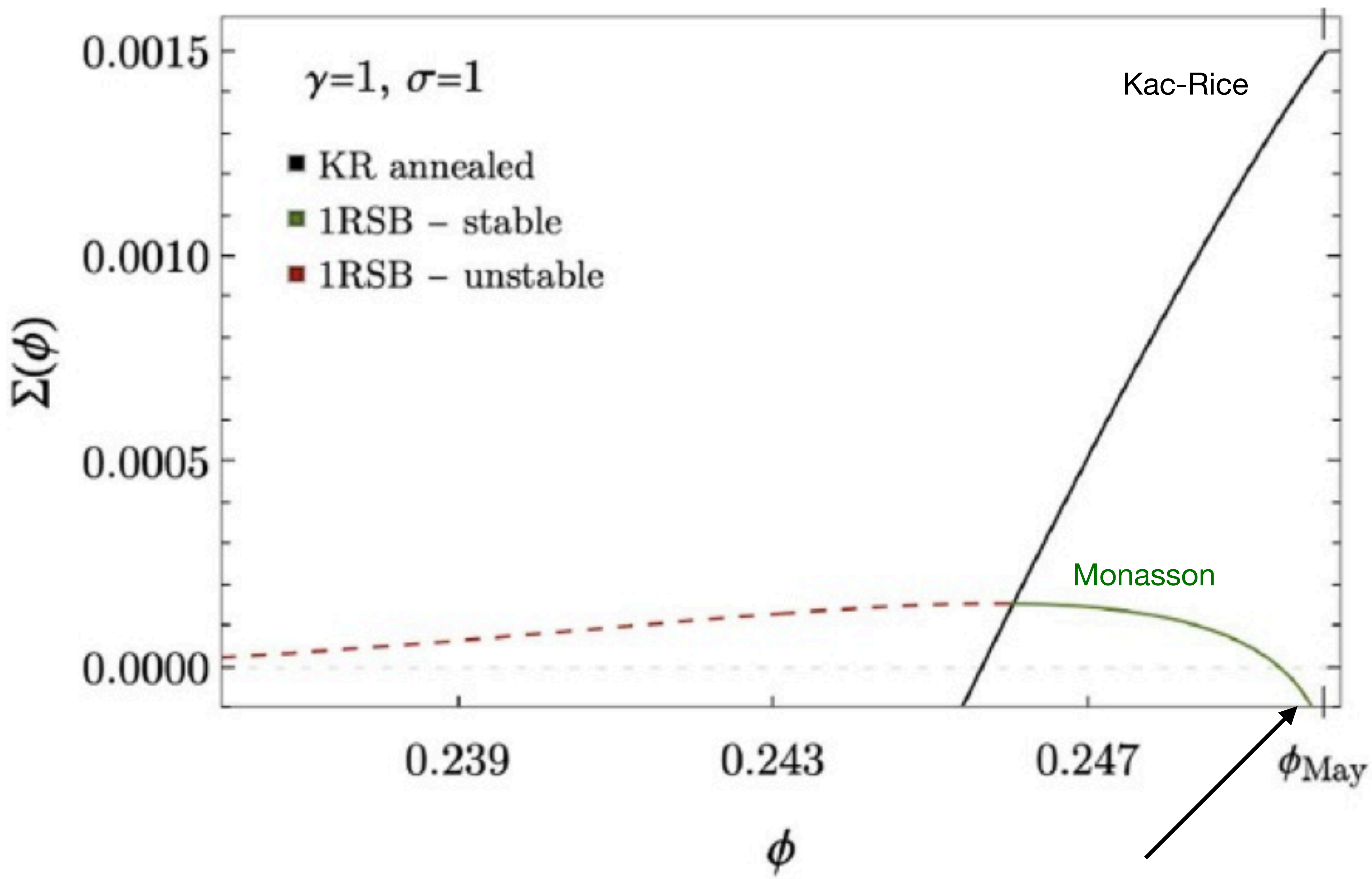


For details: VR, Roy, Biroli, Bunin & Turner, PRL 130, 257401 (2023)

VR, Roy, Biroli, Bunin, J. Phys. A 56, 305003 (2023)

General γ: ongoing (with A. Pacco)

Back to "standard" counting: a comparison



$\phi_{\mathrm{1RSB}} = 0.2494$

estimate of diversity at the GS (fullRSB needed)

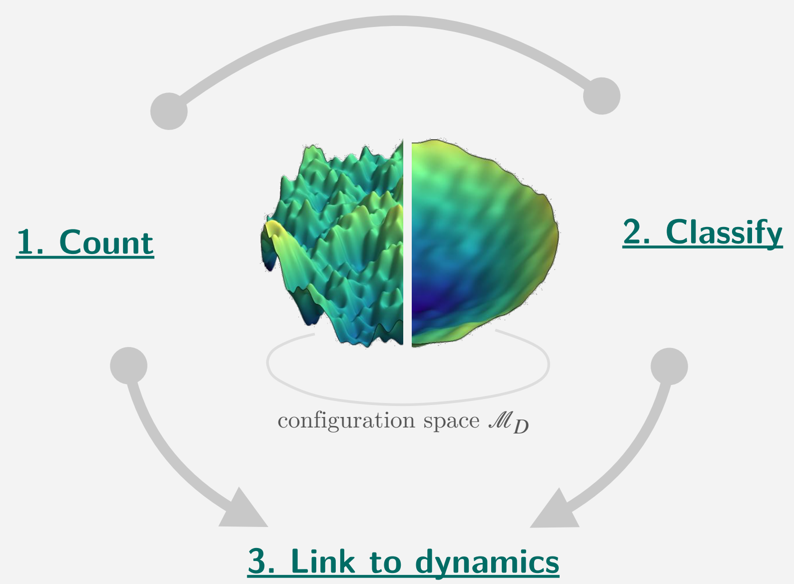
For details: VR, Roy, Biroli, Bunin & Turner, PRL 130, 257401 (2023)

VR, Roy, Biroli, Bunin, J. Phys. A 56, 305003 (2023)

Summary.

■ Multiple competing dynamical attractors/stationary points is key feature of complex (glassy) systems.

■ Characterizing their distribution is relevant for:
→ **dynamics beyond mean-field (activated)**
→ **chaos (instability) vs aging (marginality)....**



■ Recent formalism (Kac-Rice) lead to interesting problems in **Random Matrix Theory.**

A review:

VR, Fyodorov – The high-d landscapes paradigm: spin-glasses, and beyond (2023)

Saddles & activation:

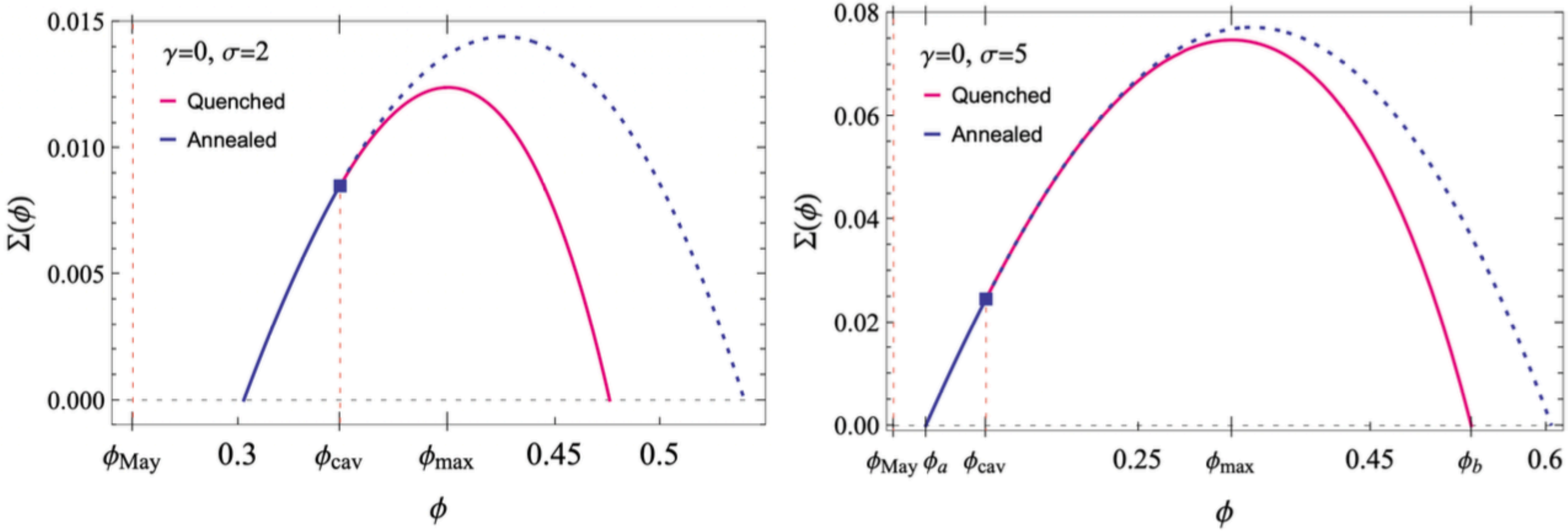
VR – Distribution of rare saddles in the p-spin energy landscape (2020)

VR, Biroli, Cammarota – Complexity of energy barriers in mean-field glassy systems (2019)

Ecosystems equilibria:

VR, Roy, Biroli, Bunin, Turner – Generalized Lotka-Volterra equations with random, non-reciprocal interactions: the typical number of equilibria (2023)

VR, Roy, Biroli, Bunin – Quenched complexity of equilibria for asymmetric Generalized Lotka-Volterra equations (2023)



γ=0, σ=2
Quenched
Annealed
0.015
0.010
0.005
0.000
Σ(ϕ)
ϕ_May
0.3
ϕ_cav
ϕ_max
0.45
0.5
ϕ
γ=0, σ=5
Quenched
Annealed
0.08
0.06
0.04
0.02
0.00
Σ(ϕ)
ϕ_May
ϕ_a
ϕ_cav
0.25
ϕ_max
0.45
ϕ_b
0.6
ϕ